

Rapid social connectivity

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Abstract

Given a graph $G = (V, E)$, consider $\text{Poisson}(|V|)$ walkers performing independent lazy simple random walks on G simultaneously, where the initial position of each walker is chosen independently w.p. proportional to the degrees. When two walkers visit the same vertex at the same time they are declared to be acquainted. The social connectivity time, $\text{SC}(G)$, is defined as the first time in which there is a path of acquaintances between every pair of walkers. It is shown that when the maximal degree of G is d , with high probability $c \log |V| \leq \text{SC}(G) \leq C_d \log^3 |V|$. We determine $\text{SC}(G)$ up to a constant factor in the cases that G is an expander or a d -dimensional torus ($d \geq 1$) of side length n .

Keywords: Social network, random walks, giant component, coalescence process.

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1 Introduction

Consider the following model for a social network which we call the **random walks social network model** or for shorts, the *social network model*. We write SN as a shorthand for social network. Let $G = (V, E)$ be a finite connected graph, which is the underlying graph of the SN model. In the model there are walkers performing independent lazy simple random walks on G , denoted **LSRW**, defined as follows. If a walker's current position is v , then the walker either stays put w.p. $1/2$, or moves to one of the neighbors of v w.p. $\frac{1}{2d_v}$, where d_v is the degree of v . The walkers perform their LSRWs simultaneously (i.e. at each time unit they all perform one step, which may be a lazy step). At time 0 the number of walkers at vertex v is set to be N_v , where $(N_v)_{v \in V}$ are independent random variables, where for each vertex v , $N_v \sim \text{Poisson}(\bar{\pi}_v)$ where $\bar{\pi} = |V|\pi$ and $\pi_v := d_v/(2|E|)$ is the stationary distribution of the random walk on G .

We say that two walkers w, w' have **met by time t** , which we denote by $w \xleftrightarrow{t} w'$, if there exists $t_0 \leq t$ such that they have the same position at time t_0 . After two walkers meet they continue their walks independently without coalescing. “Meeting by time t ” is a symmetric relation. It induces the equivalence relation of **having a path of acquaintances by time t** , denoted by $\overset{t}{\sim}$, defined as follows. Two walkers a and b have a path of acquaintances by time t iff there exist $k \in \mathbb{N}$ and walkers $a = c_0, c_1, \dots, c_k, c_{k+1} = b$ such that $c_i \xleftrightarrow{t} c_{i+1}$, for all $0 \leq i \leq k$. Note that we are not requiring the sequence of times in which the walkers met to be non-decreasing, which is the main difference between the SN model and some existing models for spread of rumor/infection (e.g. the $A + B \rightarrow 2B$ model [12] and the Frog model [18, 3, 4, 17, 10], see § 1.5 for a description of these models and a discussion about their relationship to the SN model). Consequently, the SN model often evolves faster than such models. However, Conjecture 1.9 suggests one sense in which also theses models might evolve rapidly.

We are interested in the coalescence process of the equivalence classes of $\overset{t}{\sim}$ (as t varies). In particular, we are interested in the **social connectivity time**, $\text{SC}(G)$, defined as the minimal time in which there is only one class (i.e. every pair of walkers have a path of acquaintances between them).

1.1 General graphs

Our main result is

Theorem 1. *Let $G = (V, E)$ be a connected graph of maximal degree d . Denote $r_* := \min_{u,v \in V} \pi_v/\pi_u \geq 1/d$. Then there exist constants $C_d, M, c > 0$ such that*

$$\mathbb{P}[\text{SC}(G) > C_d \log^3 |V|] \leq M/|V|. \quad (1.1)$$

$$\forall \alpha \in (0, 1), \exists c_\alpha > 0, \quad \mathbb{P}[\text{SC}(G) \leq c_\alpha \log |V|] \leq e^{-cr_*^\alpha |V|}. \quad (1.2)$$

Remark 1.1. *The constant C_d from (1.1) has a polynomial dependence in d . When G is d -regular we can take $C_d = Cd^2$ (even in this case Remark 8.8 describe a family of examples in which the true dependence on d is linear). From the proof of Theorem 1, it follows that we can replace $C_d \log^3 |V|$ by $\tilde{C}d^8 t_* \log |V|$ (resp. $\tilde{C}d^2 t_* \log |V|$ when G is regular), where t_**

is the minimal t such that $\max_{x,y} |p^t(x,y) - \pi_y| \leq 1/(12d \log |V|)$ (when G is regular, d can be omitted from the denominator). By Fact 3.3, $t_* \leq C'd^2 \log^2 |V|$ (resp. $\leq C' \log^2 |V|$, when G is regular). For the cycle $t_* = \Theta(\log^2 |V|)$. We believe that the cycle is in some sense extremal for the SN model (see Remark 8.8, Theorem 3 and Conjecture 8.3 for more details on this point).

Remark 1.2. If G is d -regular and in addition $|\{\{u,v\} \in E : u \in A, v \notin A\}| \geq c_0 d$, for every set $A \subset V$ such that $|A| \leq |V|/2$, then one can replace C_d in (1.1) by some absolute constant C (see part (iii) of Proposition 3.7).

Remark 1.3. Note that (1.2) is meaningful as long as $r_* \geq |V|^{\beta-0.5}$ for some $\beta > 0$. Theorem 4.4 gives an extension covering in particular the case $r_* \geq |V|^{\beta-1}$ for some $\beta > 0$.

Remark 1.4. One may consider the case in which the density of walkers at each site is multiplied by some $\lambda \leq 1$ (the case that $\lambda = \lambda_{|V|} = \Theta(\log |V|)$ is discussed in Remark 8.5). The proof of (1.1) can be extended to this setup and the obtained bound is decreasing in λ (for $\lambda \leq 1$ it is of the form $C_d \lambda^{-2} \log^3(\lambda |V|)$). By (1.1), if the graph is sparsely occupied so that there are still classes of walkers of size 1 at time $C_d \log^3 |V|$ (it follows from the analysis in § 4 that when G is regular and $1/|V| \ll \lambda = \lambda_{|V|} \ll 1/\log^3 |V|$, w.h.p. this is indeed the case) then the social connectivity time would be larger than for a dense community with $\lambda = 1$. When G is vertex-transitive (Definition 8.2) we Conjecture that

$$\mathbb{E}_\lambda[\text{SC}(G) \mid \text{there are at least two walkers}]$$

is monotone decreasing in λ (see Conjecture 8.4 for a slightly different formulation).

Theorem 2. There exists $C > 0$ such that for every connected regular graph $G = (V, E)$,

$$\mathbb{P}[\text{SC}(G) < \log |V| - 6 \log \log |V|] \leq C / \log |V|.$$

Remark 1.5. We will prove Theorem 2 for an arbitrary holding probability $p \in [0, 1)$ for the walks (i.e. the probability of a walker to stay put at each given step is p). Let K_n be the complete graph on n vertices. Consider the SN model on K_n with holding probability either $p_n = 0$ or $p_n = 1/n$. Then $\text{SC}(K_n)$ is concentrated around $\log n$ in a window whose width is of order 1. Thus, the complete graph is the regular graph with (asymptotically) the fastest social connectivity time (at least when the holding probability is either 0 or $1/(d+1)$). The proof is similar to the proof that the connectivity threshold for $G(n, p)$, the Erdős and Rényi random graph, occurs in a window of order $1/n$ around $p_n = (\log n)/n$ (for the SN model the proof is still elementary, but it is somewhat tedious and thus omitted).

Remark 1.6. A directed graph G is called Eulerian if for every vertex v the indegree of v equals its outdegree, denoted (unambiguously) by d_v . One can define LSRW on $G = (V, E)$ in an analogous manner to the undirected case (e.g. [6]). Its stationary distribution is again $\pi_v := d_v/(2|E|)$ (for Eulerian G). We note that Theorems 1, 2 and 4.4 and Remarks 1.1-1.2 all remain valid when G is an Eulerian digraph. The proofs require only minor adjustments. See § 8 for other extensions of our results (e.g. allowing the walkers to belong to different communities, walking on different graphs with the same vertex set).

1.2 Tori

Theorem 3. *Let C_n be the n -cycle. Consider the setup in which at time 0 there is exactly one walker at each vertex and that the walkers perform continuous-time independent SRW (with jump rate 1). Then there exist some absolute constants $c_1, c_2, \alpha, C_1, C_2 > 0$ such that*

$$\mathbb{P}[\text{SC}(C_n) > C_1 \log^2 n] \leq C_2/n, \quad (1.3)$$

$$\mathbb{P}[\text{SC}(C_n) < c_1 \log^2 n] \leq e^{-c_2 n^\alpha}. \quad (1.4)$$

The proof of (1.3) is due to Gady Kozma. We thank him for allowing us to present his argument.

Remark 1.7. *Working in continuous-time simplifies the analysis of (1.3). The proof in the setup of $\text{Pois}(1)$ walkers per site is in fact simpler (see Remark 5.1).*

We denote the d -dimensional torus of side length n by $\mathbb{T}_d(n)$. This is the Cayley graph of $(\mathbb{Z}/n\mathbb{Z})^d$ obtained by connecting each $x, y \in (\mathbb{Z}/n\mathbb{Z})^d$ which disagree only in one coordinate, by $\pm 1 \bmod n$. We refer to a class of walkers (in the SN model on some graph $G = (V, E)$ at a certain time) of size $\Omega(|V|)$ as a ***giant class***. Part (i) of Theorem 4 asserts that for $\mathbb{T}_d(n)$ ($d \geq 2$) a giant class emerges in constant time w.h.p..

Theorem 4. *Consider the setup in which the walkers perform independent rate 1 continuous-time SRW on $\mathbb{T}_d(n)$. Let $M_t(\mathbb{T}_d(n))$ be the size of the largest class of walkers at time t .*

- (i) *There exist some t_d, c_d such that $\lim_{n \rightarrow \infty} \mathbb{P}[M_{t_d}(\mathbb{T}_d(n)) \geq c_d n^d] = 1$, for all $d \geq 2$.*
- (ii) *There exist $c, C > 0$ such that w.h.p. $c \leq \text{SC}(\mathbb{T}_2(n))/(\log n \log \log n) \leq C$.*
- (iii) *For all $d > 2$ there exists $C'_d > 0$ such that w.h.p. $0.99 \leq \text{SC}(\mathbb{T}_d(n))/\log n \leq C'_d$.*

1.3 Expanders

We denote by $\lambda(G)$ the ***spectral-gap*** of LSRW on G , defined as the smallest non-zero eigenvalue of $I - P$, where P is the transition matrix of LSRW on G . We say that a sequence of graphs G_n is an expander family if $\inf_n \lambda(G_n) > 0$. We say that G is a ***λ -expander*** if $\lambda(G) \geq \lambda$ (we think of λ as being uniformly bounded away from 0, independently of the size of G , although our analysis remains valid even if this fails). The following result demonstrates that Theorem 2 is sharp.

Theorem 5. *Let G be a connected d -regular n -vertex λ -expander. Then there exist absolute constants C, C', C_0 such that if $\lambda \log_d n \geq C_0$, then $\text{SC}(G) \leq C \lambda^{-1} \log n$ w.p. at least $1 - C'/n$.*

Theorem 6. *Let G be a connected d -regular n -vertex λ -expander. Then there exist some constants $c, C_1 > 0$ such that with probability at least $1 - e^{-cnd^{-4(s+1)}}$, after $s := \lceil C_1/\lambda \rceil$ steps there exists a class of walkers (all having a path of acquaintances between them by time s) of size at least $n/6$.*

Remark 1.8. We shall deduce Theorem 5 from Theorem 6 by bounding SC by the time it takes the giant class from Theorem 6 to “capture” the rest of the walkers. The requirement $\lambda \log_d n \geq C_0$ in Theorem 5 is imposed in order to ensure that the error term $e^{-cnd^{-4(s+1)}}$ from Theorem 6 is small. We believe that a more careful analysis can allow one to replace the error term $e^{-cnd^{-4(s+1)}}$ (which is obtained due to a rather naive application of Azuma inequality in the proof of Corollary 6.4) by some other error term which is independent of d . This would extend the applicability of Theorems 5 and 6 by allowing d to be arbitrary.

An alternative approach for eliminating the dependence on d is to establish the existence of a giant class after a constant number of steps, using the exploration process from the proofs of [9, Theorem 3] and [5, Theorem 3] (and combining ideas from these proofs).

1.4 Organization of this work

In § 2 we present some preliminaries. In § 3 we prove (1.1). In § 3 we prove (1.2) and Theorem 2. In § 5 we prove Theorem 3. In § 6 we study the SN model on expanders and prove Theorems 5-6. In § 7 we prove Theorem 4. We conclude with some open problems and conjectures and concluding remarks, § 8.

1.5 Related work and connections to other models

The social network model on infinite graphs is investigated in [9]. The setup studied in [9] is as follows. Given an infinite connected regular graph $G = (V, E)$, place at each vertex $\text{Poisson}(\lambda)$ walkers performing independent lazy simple random walks on G simultaneously. When two walkers visit the same vertex at the same time they are declared to be acquainted. It is shown in [9] that when G is vertex-transitive and amenable, for all $\lambda > 0$ a.s. every pair of walkers will eventually have a path of acquaintances between them. In contrast, it is shown that when G is non-amenable (not necessarily transitive) there is always a phase transition at some $\lambda_c(G) > 0$. General bounds on $\lambda_c(G)$ are given and the case that G is the d -regular tree, \mathcal{T}_d , is studied in more details (it is shown that $c \leq \lambda_c(\mathcal{T}_d)/\sqrt{d} \leq C$). Finally, it is shown that in the non-amenable setup there exists a finite time t such that a.s. there exists an infinite set of walkers having a path of acquaintances between them by time t . We note that the last result is the infinite setup counterpart of our Theorem 6.

We believe that, in spirit, the results in this paper should be true also for some other particle systems involving independent random walks. The $A + B \rightarrow 2B$ family of models (e.g. [12, 13]), often interpreted as models for a spread of an epidemic/rumor, are defined by the following rule: there are type A and B particles occupying a graph G , say with densities $\lambda_A, \lambda_B > 0$. They perform independent SRW with holding probabilities $p_A \in [0, 1]$ and $p_B \in [0, 1]$. When a type B particle collides with a type A particle, the latter transforms into a type B particle. The *frog model* is the particular case of the above dynamics in which the type A particles are immobile ($p_A = 1$) and $\lambda_A = \lambda_B = \lambda$.

We now consider a variant of the above model which is intimately related to the SN model. Consider the case in which initially only the particles at some vertex o are of type B and an additional B particle is planted at o (this is done to ensure that there is at least one B particle). Consider the case that each B particle performs only t steps of its walk before vanishing (after which it cannot become reactivated again). We call t the lifetime of the B

particles. We say that the process dies out when there are no B particles left. One then defines the *susceptibility*, $\mathcal{S}(G)$ as the minimal lifetime which is sufficient so that all particles are transformed into part B particles before the process dies out (to make this definition formal, we think of the particles as first picking the infinite trajectory of their random walks, but then, once transforming into a B particle, performing only t additional steps of the walk). In the interpretation of the model as a model for a spread of rumor/epidemic the susceptibility is meant to capture “how long should a virus live in order to infect the entire population” or “how interesting should a rumor be, so that eventually everybody will hear it”.

In [8, 5] parallel results to Theorems 2,3,4,5 and 6 are proved about the above variant in the case of the frog model, where the susceptibility replaces the social connectivity time. We strongly believe that (certain variants of) Theorems 2,3,4,5 and 6 should hold even when the A particles are mobile. Moreover, we have the following conjecture concerning an analog of Theorem 1.

Conjecture 1.9. *Consider the aforementioned variant of the $A + B \rightarrow 2B$ model with $\lambda_A = \lambda_B = \lambda$ and some fixed $p_A \in [0, 1]$ and $p_B \in (0, 1)$. Then there exist some $C_{d,\lambda}, \ell > 0$, such that for every sequence of finite connected graphs $G_n = (V_n, E_n)$ with $|V_n| \rightarrow \infty$ of maximal degree at most d , we have that $\lim_{n \rightarrow \infty} \mathbb{P}_\lambda[\mathcal{S}(G_n) \leq C_{d,\lambda} \log^\ell |V_n|] = 1$.*

2 Preliminaries and additional notation

2.1 Reversibility, stationarity of the occupation measure and independence of the number of walkers performing different walks.

Let $G = (V, E)$ be a graph. Let P be the transition kernel of LSRW on G . We denote by $p^t(u, v) := P^t(u, v)$ the t -steps transition probability from u to v . We now establish a certain independence property for walks in G , which in particular implies a certain stationarity property of the SN model. Recall that $\pi_v := d_v/2|E|$ and $\bar{\pi} = |V|\pi$. For LSRW on G , **reversibility** is the property that for every $v, u \in V$ and $t \geq 0$, $\pi_v p^t(v, u) = \pi_u p^t(u, v)$. A **walk** of length k in G is a sequence of $k+1$ vertices (v_0, v_1, \dots, v_k) such that for all $0 \leq i < k$ either $v_i = v_{i+1}$ or $\{v_i, v_{i+1}\} \in E$. Let Γ_k be the collection of all walks of length k in G .

For a walk $\gamma = (\gamma_0, \dots, \gamma_k) \in \Gamma_k$ for some $k \geq 1$, we denote $p(\gamma) := \prod_{i=0}^{k-1} P(\gamma_i, \gamma_{i+1})$ and $q(\gamma) := \bar{\pi}_{\gamma_0} p(\gamma)$. Note that $p(\gamma)$ is precisely the probability that a walker whose starting position is γ_0 performed the walk γ (i.e. her position at time i is γ_i for all $i \leq k$). Let γ' be the reversal of γ (i.e. $\gamma'_i = \gamma_{|\gamma|-i}$ for all $i \leq |\gamma|$). Reversibility is equivalent to the following:

$$q(\gamma) = q(\gamma'), \text{ for every walk } \gamma.$$

We denote the number of walkers who performed a walk γ by X_γ . We denote the number of walkers whose position at time t is v by $Y_v(t)$. Since π is the stationary distribution of the walks, for every vertex v and time $t > 0$, $\mathbb{E}[Y_v(t)] = \sum_{u \in V} \bar{\pi}_u p^t(u, v) = \bar{\pi}_v = \mathbb{E}[Y_v(0)]$. Thus, the following fact follows easily from Poisson thinning.

Fact 2.1. *Let G be a connected graph. For all $t > 0$ and $\gamma \in \Gamma_t$, $X_\gamma \sim \text{Poisson}(q(\gamma))$. Moreover, for each fixed $t > 0$, $(X_\gamma)_{\gamma \in \Gamma_t}$ are independent. Consequently, for each fixed $t > 0$, $(Y_v(t))_{v \in V}$ are independent and $Y_v(t) \sim \text{Poisson}(\bar{\pi}_v)$ for all $v \in V$.*

Next we note that if the maximal degree of G is d , then for all $v, u \in V$

$$1/d \leq \bar{\pi}_v \leq d, \quad 1/d \leq \bar{\pi}_v/\bar{\pi}_u \leq d. \quad (2.1)$$

Lastly, we need the following concentration estimate. Let $Y \sim \text{Pois}(\mu)$. Then for all $a \geq 0$

$$\mathbb{P}[Y \geq \mu + a] \leq \exp\left(-\frac{a^2/\mu}{2(1 + a/(3\mu))}\right). \quad (2.2)$$

3 A general upper bound on SC - proof of (1.1)

Consider an arbitrary discrete-time coalescent process starting with n distinct classes. Define the coalescence time, CT, as the minimal time in which there is only one class (if there is no such time it is defined to be ∞). We now describe several simple conditions, each of which leads to an upper bound on the coalescence time. We intentionally start with two simple deterministic conditions and then work our way towards a slightly more complicated condition which is more faithful to the actual details of our argument.

Clearly, if prior to CT in every time unit every class merges with at least one other class, then deterministically, it must be the case that $\text{CT} \leq \lceil \log_2 n \rceil$. Fix some $t > 0$ and denote $t_k = kt$ we call the time interval $(t_k, t_{k+1}]$ the $(k+1)$ -**th round**. We say that the $k+1$ -th round is a **p -merging round** if either $t_k \geq \text{CT}$ (this case is taken for technical reasons), or $t_k < \text{CT}$ and at least a p -fraction of the classes (w.r.t. time t_k) have merged with at least one other class by time t_{k+1} (including). Note that if the number of classes at the beginning of a p -merging round was $m > 1$, then the number of classes at the end of the round is at most $m(1 - p/2)$. If all rounds are p -merging then, deterministically, $\text{CT} \leq \lceil C_p t \log |V| \rceil$ (for concreteness, $C_p := 1/\log(\frac{2}{2-p})$). More generally, even if not all rounds are p -merging, there can be at most $\lceil C_p \log |V| \rceil$ p -merging rounds prior to CT. This motivates the following simple model in which each round is p -merging with probability at least α , regardless of the history. We intentionally keep the description and analysis of this model somewhat informal, deferring a more rigorous statement to Lemma 3.1.

Consider now the following “baby coalescent model”: start with n classes. At each round flip a coin with heads probability at least α (i.e. the heads probability in each round may depend on the history up to that time, but is guaranteed to be at least α). If it lands on heads then the round is p -merging. The list of pairs of classes which merge with one another in each round (given the evolution of the model up to the beginning of the round and the result of the coin flip corresponding to the round) is determined according to some arbitrary predetermined rule (satisfying that whenever the coin lands on heads the requirement of being p -merging is satisfied by the list of pairs of merged classes), perhaps using some additional independent source of randomness. The predetermined rule can be thought of as one decided by an adversary.

It is intuitively clear that there exists $C_{\alpha,p}$ (sufficiently large in terms of C_p/α , where C_p is as above) such that the probability that the total number of rounds exceeds $L = L_{n,\alpha,p} := \lceil C_{\alpha,p} \log n \rceil$ tends to 0 as $n \rightarrow \infty$. Informally, one can argue that the number of p -merging rounds in the first L rounds, stochastically dominates the $\text{Bin}(L, \alpha)$ distribution and then apply an appropriate large deviation estimate in order to bound $\mathbb{P}[\text{Bin}(L, \alpha) < C_p \log n]$.

Moreover, if at each round we only have probability $\geq 1 - \delta_n$, for some $\delta_n = o(1/\log n)$, that the model evolves according to the above description, then it is intuitively clear that the above conclusion remains valid (with an additional error term of $\delta_n L = o(1)$). Note that $C_{\alpha,p} \leq K/(\alpha p)$ for some absolute constant K .

Conceptually, we will show that the analysis of the SN model on a graph $G = (V, E)$ of maximal degree d can be reduced to that of the modification of the “baby model” involving the δ_n term (for some $t = t_{d,|V|} = \lceil C_d \log^2 |V| \rceil$, $\alpha = \alpha_d = p$, $\delta_n = |V|^{-3}$ and some random $n \leq |V|$; When G is regular $t = t_{|V|} = \lceil C \log^2 |V| \rceil$, $\alpha = c/d = p$, and under the assumption from part (iii) of Proposition 3.7, $t = \lceil C \log^2 |V| \rceil$, $\alpha = c = p$).

Observe that above it was sufficient to consider the number of classes at the end of each round in order to bound CT from above. This motivates the following lemma, which formulates rigorously the assertions made in the previous paragraph (think about $J_k + 1$ below as the number of classes at the end of the k -th round, and so $T_0 := \inf\{k : J_k = 0\}$ corresponds to CT).

Lemma 3.1. *Let J_k be a non-increasing \mathbb{Z}_+ -valued random process, measurable w.r.t. filtration $(\mathcal{F}_k)_{k \geq 0}$, with $J_0 \leq n$, satisfying that for some $0 < \alpha, p, \delta_n < 1$, for each k there exists some $A_k \in \mathcal{F}_k$ with $\mathbb{P}(A_k) \geq 1 - \delta_n$ satisfying that*

$$\mathbb{E}[1_{\{J_{k+1} \leq (1-p/2)J_k\}} 1_{A_k} \mid \mathcal{F}_k] \geq \alpha 1_{A_k} \quad (3.1)$$

Let $T_0 := \inf\{k : J_k = 0\}$. Then there exists some $C_{\alpha,p} \leq K/(\alpha p)$ such that

$$\mathbb{P}[T_0 > \lceil C_{\alpha,p} \log n \rceil] \leq n^{-2} + \delta_n (\lceil C_{\alpha,p} \log n \rceil + 1).$$

Proof: Denote the complement of A_k by B_k . Let $\tau := \inf\{k : 1_{B_k} = 1\}$. Define $I_k = J_k$ for $k < \tau$ and $I_k = 0$ for $k \geq \tau$. Denote $L := \lceil C_{\alpha,p} \log n \rceil$, where $C_{\alpha,p}$ shall be determined soon. Let $T'_0 := \inf\{k : I_k = 0\}$. By a union bound over $\bigcup_{i=0}^L B_k$ it suffices to show that $\mathbb{P}[T'_0 \geq L] \leq n^{-2}$ (since $\mathbb{P}[T_0 > L] - \mathbb{P}[T'_0 > L] \leq \mathbb{P}(\bigcup_{i=0}^L B_k)$). Now, by (3.1) $M_k := a^k I_k$ is a super-martingale (w.r.t. the filtration $(\mathcal{F}_k)_{k \geq 0}$), where $a := 1/(1 - \frac{1}{2}\alpha p)$. In particular, for $C_{\alpha,p} := 3/\log a$ we have $\mathbb{E}[I_L] \leq a^{-L} \mathbb{E}[M_0] \leq n^{-3} \mathbb{E}[J_0] \leq n^{-2}$. By Markov inequality

$$\mathbb{P}[T'_0 \geq L] = \mathbb{P}[I_L \geq 1] \leq \mathbb{E}[I_L] \leq n^{-2}. \quad \square$$

We are interested in establishing a similar behavior for the SN model. Recall, that $Y_v(s)$ is the number of walkers at vertex v at time s . It is beneficial to consider a notion of “independence of the history” which depends only on $(Y_v(s))_{v \in V}$ (whose distribution is stationary and hence convenient to work with), rather than also on the identity of the classes of walkers at time s .

Fix some connected graph $G = (V, E)$ of maximal degree d . Set $t = t_{d,|V|} := \lceil C_d \log^2 |V| \rceil$ for some sufficiently large constant C_d , to be determined later (t shall remain fixed until the end of the section; for regular G set $t := \lceil C \log^2 |V| \rceil$). As before, we denote $t_k := kt$ and call the time between t_k and t_{k+1} the $(k+1)$ -th **round**.

We say that a partition of the walkers at some time s into (disjoint) sets **respects vertices** if walkers which are at the same vertex at time s all belong to the same set.

We say that the configuration $(Y_v(s))_{v \in V}$ at time s has the α -**merging property**, if the following holds. For every vertex-respecting partition of the walkers into sets, $\mathcal{A}_1, \dots, \mathcal{A}_m$, with $m \geq 2$, for every \mathcal{A}_i which contains less than half of the walkers, the probability that at least one walker from \mathcal{A}_i will meet some walker not belonging to \mathcal{A}_i in the time interval $(s, s + t]$ is at least α (t is as above, in our application we shall consider $s = t_k$ for some k , and then $s + t = t_{k+1}$). This definition depends on t . It would have been appropriate to use the term “the α -merging property in t steps”, but since t , although not a constant (it depends on $|V|$), is fixed, we suppress the dependence on t .

The condition in the definition is seemingly strong (and thus, a-priori, hard to verify), as the sets $\mathcal{A}_1, \dots, \mathcal{A}_m$ form an arbitrary vertex-respecting partition of the walkers into disjoint sets, which need not be related to the identity of the classes of walkers. However, we shall see (Proposition 3.7) that in fact this condition follows from another rather simple condition on the configuration of walkers, which holds for each fixed time w.p. at least $1 - |V|^{-3}$.

One sense in which the condition is seemingly weaker than the situation we had in the “baby model” (in which p -merging rounds were considered) and in Lemma 3.1 (in which J_k , decoding the number of classes minus one at the end of round k , dropped by a constant factor in some rounds) is that here we only require a uniform lower bound on the marginal probability that at least one walker from a given set of walkers would meet in t steps (i.e. in a given round) a walker not from this set. However, as the following lemma asserts, a uniform lower bound on marginal probabilities of certain m events, implies a corresponding lower bound on the probability that a certain fraction of them occur. The lemma follows by a straightforward application of the Paley-Zygmund inequality (e.g. [11, Lemma 4.1]).

Lemma 3.2. *Let ξ_1, \dots, ξ_m be indicator random variables. Denote $p_i := \mathbb{E}[\xi_i]$, $p := \min_i p_i$, $S := \sum_i \xi_i$ and $\mu = \mathbb{E}[S]$. Then*

$$\mathbb{P}[S \geq pm/2] \geq \mathbb{P}[S \geq \mu/2] \geq (1/2)^2 \mu^2 / \mathbb{E}[S^2] \geq \mu / (4m) \geq p/4. \quad (3.2)$$

In our setup this implies the following. Assuming that the configuration at time s has the α -merging property and $\mathcal{A}_1, \dots, \mathcal{A}_m$ is a vertex-respecting partition of the walkers, then with probability at least $\alpha/4$, there exists an index set I of size at least $\lceil \alpha(m-1)/2 \rceil$ such that for all $i \in I$, there is some walker from \mathcal{A}_i who met in the time interval $(s, s + t]$ some walker not belonging to \mathcal{A}_i . This explains why in the last paragraph of the page 6 we argued that we can take $\alpha = p$ (by decreasing one of them by a factor of 2).

Crucially, note that the notion of having the merging property (with some fixed parameter) depends only on the current configuration of walkers. In particular, it is independent of the identity of the classes at the relevant time and also of the time itself. By Lemmas 3.1-3.2, if we knew that for some α , the SN model has the α -merging property at each fixed time w.p. at least $1 - 1/|V|^2$, then we would get that there exists some C'_d such that $\text{SC}(G) \leq C'_d t \log |V|$ w.p. at least $1 - (2 + C'_d \log |V|)/|V|^2$. Since C'_d have a polynomial dependence on d , say $C'_d \leq C' d^\ell$, and by increasing some of the absolute constants we can replace $|V|^2$ in the denominator above by $|V|^{\ell+2}$, we get that $\text{SC}(G) \leq \tilde{C}_d t \log |V|$ w.p. at least $1 - (C' \log |V|)/|V|^2$.

To conclude the proof of the upper bound on $\text{SC}(G)$, we now consider another property of a configuration of walkers (which is seemingly simpler than the merging property) and

argue that it is satisfied at every fixed time w.p. at least $1 - 1/|V|^3$ (assuming C_d and $|V|$ are sufficiently large) and that it, deterministically, implies the merging property (with some related parameter).

Let t be as above and $0 < \delta < 1/8$. We say that the configuration $(Y_v(s))_{v \in V}$ is **δ -balanced** (or “ δ -balanced in t steps”, but below we suppress the t dependence), if

$$\forall v \in V, \quad \mathbb{E}[Y_v(s+t) \mid (Y_u(s))_{u \in V}] = \sum_u p^t(u, v) Y_u(s) \geq (1 - \delta) \bar{\pi}_v \quad (3.3)$$

(recall $\bar{\pi}_v := |V| \pi_v$). In our application we shall take δ to be some fixed constant, independent of d (in fact, any fixed $\delta \in (0, 1/8)$ works, and with some extra care, also any fixed $\delta \in (0, 1/2)$ works), so we suppress the dependence on δ of certain quantities in our notation below.

Note that if $(Y_v(s))_{v \in V}$ is distributed like the configuration of walkers in the SN model on G at time s , then (by stationarity of the occupation measure, which allows one to consider only the case that $s = 0$) for every $v \in V$ the sum $\bar{Z}_v(s) := \sum_u p^t(u, v) Y_u(s)$ in (3.3) (whose mean is $\bar{\pi}_v$) is a linear combination (with non-negative coefficients) of independent Poisson r.v.’s. As demonstrated in Lemma 3.5 in order to obtain a large deviation estimate for such a linear combination (namely, for the event $\bar{Z}_v(s) < (1 - \delta) \bar{\pi}_v = (1 - \delta) \mathbb{E}[\bar{Z}_v(s)]$), it suffices to control the maximal coefficient appearing in the sum. We obtain such control using the following decay estimate for $\max_{x,y} p^t(x, y)$.

Fact 3.3 ([6] Lemma 2.4). *There exists an absolute constant M such that for every finite connected graph (or Eulerian digraph) $G = (V, E)$ LSRW on G satisfies*

$$\forall t \geq 1, \quad \max_{x,y} |p^t(x, y) - \pi_y| \leq M \deg(y) \sqrt{1/t}. \quad (3.4)$$

Moreover, if G is regular, then

$$\forall t \geq 1, \quad \max_{x,y} |p^t(x, y) - \pi_y| \leq M \sqrt{1/t}. \quad (3.5)$$

In what comes, we shall assume that if G is not regular, then (*) $12d^2 \log |V| \leq \delta^2 |V|$. The following proposition, whose proof is deferred to § 3.1 covers the case that (*) fails.

Proposition 3.4. *Let $G = (V, E)$ be a connected simple graph. Then for some $C > 0$,*

$$\mathbb{P}[\text{SC}(G) > C|E||V| \log |V|] \leq C/|V|.$$

By Fact 3.3, for all $c > 0$, if we set $C_d = C_d(c) = (Md/c)^2$ (resp. for regular G , we set $t := \lceil (M/c)^2 \log^2 |V| \rceil$) then

$$\max_{x,y} p^t(x, y) \leq \pi_y + c / \log |V|. \quad (3.6)$$

Assume (*) holds for some fixed $0 < \delta < 1/8$. Set $c = c_d = \delta^2 / 12d$ (i.e. we set $C_d = 144M^2 d^4 \delta^{-4}$). By (3.6) we have that

$$\max_{x,y} p^t(x, y) \leq \delta^2 / (6d \log |V|). \quad (3.7)$$

If G is regular, set $c = \frac{\delta^2}{12}$, (i.e. $t = 144\delta^{-4}M^2 \log^2 |V|$). If $(**)$ $12 \log |V| \leq \delta^2 |V|$, then

$$\max_{x,y} p^t(x,y) \leq \delta^2 / (6 \log |V|) \quad (3.8)$$

(we may assume $(**)$ holds, as otherwise, $|V| \leq \bar{C}$ and then Proposition 3.4 implies (1.1)). In light of the paragraph following (3.3), by combining (3.7) and (3.8) with the following lemma, it follows (by a union bound over the vertices of G) that the probability that the configuration is not δ -balanced at some fixed time s is at most $1/|V|^2$ (assuming $(*)$ or in the regular case assuming $(**)$, when the constants are chosen as indicated above).

Lemma 3.5. *Let ξ_1, \dots, ξ_m be independent Poisson random variables. Let $p_1, \dots, p_m \in (0, 1)$. Denote $p_* := \max p_i$, $S := \sum_{i=1}^m p_i \xi_i$ and $\mu := \mathbb{E}[S]$. Then for all $\delta \in (0, 1)$*

$$\mathbb{P}[S \leq (1 - \delta)\mu] \leq e^{-\mu\delta^2/(2p_*)}. \quad (3.9)$$

Proof: Denote $\mu_i := \mathbb{E}[\xi_i]$ and $\lambda := \delta/(2p_*)$. As $\sum_k e^{-\lambda p_i k} \mu_i^k / k! = e^{\mu_i e^{-\lambda p_i}}$ and $\lambda p_i < 1$

$$\mathbb{E}[e^{-\lambda p_i \xi_i}] = \exp[\mu_i(e^{-\lambda p_i} - 1)] \leq \exp[\mu_i(-\lambda p_i + (\lambda p_i)^2/2)] \leq \exp[\mu_i(-\lambda p_i + \lambda^2 p_* p_i/2)],$$

$$\mathbb{E}[e^{-\lambda S}] = \prod_{i=1}^m \mathbb{E}[e^{-\lambda p_i \xi_i}] \leq \exp\left[\sum_{i=1}^m \mu_i(-\lambda p_i + \lambda^2 p_* p_i/2)\right] = \exp[\mu(-\lambda + \lambda^2 p_*/2)].$$

Finally, by Markov inequality and the choice of λ

$$\mathbb{P}[S \leq (1 - \delta)\mu] = \mathbb{P}[e^{-\lambda S} \geq e^{-\lambda(1-\delta)\mu}] \leq \mathbb{E}[e^{-\lambda S}] e^{\lambda(1-\delta)\mu} \leq e^{\mu(-\lambda\delta + \lambda^2 p_*/2)} = e^{-\mu\delta^2/(2p_*)}. \quad \square$$

Before concluding the proof of (1.1) we need one more lemma.

Lemma 3.6. *The probability that the total number of walkers is at least $|V| + |V|^{2/3}$ is at most $\exp(-\frac{1}{3}|V|^{1/3})$. Moreover, if G is regular, then for every fixed time s , the probability that there exists a set $A \subset V$ of size at least $|V|/2$ such that at time s : (i) every $a \in A$ is occupied by at least one walker and (ii) A is occupied by at most $\frac{1}{2}(|V| + |V|^{2/3})$ walkers, is at most $\mathbb{P}[\text{Bin}(|V|, e^{-1}) \geq \frac{1}{2}(|V| - |V|^{2/3})] \leq e^{-c|V|}$ for some absolute constant $c > 0$.*

Proof. The bound $\exp(-\frac{1}{3}|V|^{1/3})$ from the first sentence is obtained via (2.2). For the second claim, note that if the number of vertices which are occupied by exactly one walker at time s is smaller than $\frac{1}{2}(|V| - |V|^{2/3})$, then deterministically, there cannot be such A satisfying (i) and (ii) above. The number of such vertices has the $\text{Bin}(|V|, \mathbb{P}[\text{Pois}(1) = 1]) = \text{Bin}(|V|, e^{-1})$ distribution. \square

The following proposition concludes the proof of the upper bound on $\text{SC}(G)$.

Proposition 3.7. *Let $G = (V, E)$ be a connected graph of maximal degree d . Let $0 < \delta < 1/8$. Let $t := \lceil 144M^2 d^4 \delta^{-4} \log^2 |V| \rceil$. Assume that a configuration of walkers is δ -balanced and contains a total number of walkers smaller than $|V| + |V|^{2/3}$. Then the following holds*

(i) *The configuration has the α -merging property for some $\alpha = \alpha_d = \bar{c}/d^4$.*

(ii) If G is regular and $t := \lceil 144M^2\delta^{-4}\log^2|V| \rceil$, then the configuration has the α -merging property for some $\alpha = \alpha_d = \bar{c}/d$.

(iii) In the setup of (ii), if: (a) $|\{\{u, v\} \in E : u \in A, v \notin A\}| \geq c_0 d$ for every set $A \subset V$ such that $|A| \leq |V|/2$ and (b) there does not exist a set A of size at least $|V|/2$ satisfying (i) and (ii) from Lemma 3.6, then the configuration has the α -merging property for $\alpha = \bar{c}$.

Proof. We start with parts (i)-(ii). Consider some configuration $(X_v)_{v \in V}$ of walkers (i.e. for all v the number of walkers at vertex v is X_v) with at most $|V| + |V|^{2/3}$ walkers (i.e. $\sum_v X_v \leq |V| + |V|^{2/3}$) and a vertex-respecting partition of the walkers into disjoint sets $\mathcal{A}_1, \dots, \mathcal{A}_m$. Since both the properties of a configuration which we consider (α -merging and δ -balanced) are independent of the time, w.l.o.g. we may assume that this configuration corresponds to time 0.

We now fix some $1 \leq i \leq m$ such that $|\mathcal{A}_i| \leq (|V| + |V|^{2/3})/2$ and show that \mathcal{A}_i does not violate the α_d -merging property for α_d as in (i) or (ii), resp.. Denote the collection of vertices occupied by the walkers from \mathcal{A}_j (at time 0) by B_j ($1 \leq j \leq m$). Denote

$$\mu_v(j) := \sum_{u \in B_j} p^t(u, v) X_u.$$

This is the expected number of walkers from \mathcal{A}_j at vertex v at time t . As $\sum_v \mu_v(j) = |\mathcal{A}_j|$, there can be at most one set \mathcal{A}_j with $\mu_v(j) > (1 - 2\delta)\bar{\pi}_v$ for all v (assuming $|V| \geq 8$). Hence we only have to treat the case that $\mu_v(i) \leq (1 - 2\delta)\bar{\pi}_v$ for all v and the case that $\mu_v(i) \leq (1 - 2\delta)\bar{\pi}_v$ holds for some of the vertices, but not for all.

Assume that $\mu_v(i) \leq (1 - 2\delta)\bar{\pi}_v$ for all v . Then since the configuration is δ -balanced

$$\forall v, \quad a_v(i) := \sum_{j:j \neq i} \mu_v(j) \geq (1 - \delta)\bar{\pi}_v - \mu_v(i) \geq \delta\bar{\pi}_v \geq \delta/d \quad (\text{resp. } \geq \delta \text{ for regular } G). \quad (3.10)$$

By Lemma 3.8, we get that for all v , the probability that there is a walker not from \mathcal{A}_i at v at time t is at least $1 - e^{-a_v(i)}$. Hence trivially, every walker from the set \mathcal{A}_i has a chance of at least $1 - e^{-a_v(i)}$ of meeting some walker not from \mathcal{A}_i at time t . In this case, the proof is concluded using (3.10).

Finally, assume that $\mu_v(i) > (1 - 2\delta)\bar{\pi}_v$ for some v but not for all v . We argue that in this case, there must be some vertex u such that for some absolute constant $c_1 > 0$

$$c_1/d^3 \leq c_1\bar{\pi}_u/d^2 \leq \mu_u(i) \leq (1 - 2\delta)\bar{\pi}_u \quad (\text{resp. } c_1/d \leq \mu_u(i) \leq (1 - 2\delta) \text{ for regular } G). \quad (3.11)$$

Once this is established, the proof is concluded as follows. Let u be as above. Since the configuration is δ -balanced we have that

$$a_u(i) := \sum_{j:j \neq i} \mu_u(j) \geq (1 - \delta)\bar{\pi}_u - \mu_u(i).$$

By (3.11) together with the elementary Lemma 3.8, the probability that u is occupied at time t by both a walker from \mathcal{A}_i and by a walker not from \mathcal{A}_i is at least

$$(1 - e^{-\mu_u(i)})(1 - e^{-a_u(i)}) \geq \bar{c}/d^4 \quad (\text{resp. } \geq \bar{c}/d \text{ for regular } G), \quad (3.12)$$

as desired. We now verify the existence of such u satisfying (3.11). Let

$$D_i := \{v : \mu_v(i) \geq (1 - 2\delta)\bar{\pi}_v\}.$$

We argue that if $u \notin D_i$ and $\{u, v\} \in E$ for some $v \in D_i$, then u satisfies (3.11) (by our assumption on \mathcal{A}_i , D_i is non-empty and is a proper subset of V). Here we rely on laziness in a crucial manner. The idea is that every walk γ of length t with $\gamma(0) \in B_i$ and $\gamma(t) = v \in D_i$ (that makes at least one lazy step, i.e. $\gamma(s) = \gamma(s+1)$ for some s) can be transformed into (typically) many walks which start at $\gamma(0)$ and end at $u \notin D_i$, where u some arbitrary neighbor of v , belonging to complement of D_i). Namely, these walks are obtained by making one less lazy step and adding to the walk, as its last step, a step from v to u . For a walk γ let γ' be its projection to its non-lazy version (obtained by omitting from γ coordinates with the same value as that of the previous time unit).

For every non-lazy walk γ' (of length $\leq t$) let $W_{\gamma'}$ be the collection of walks of length t whose non-lazy version is γ' . We first note that the contribution to $\mu_v(i)$ coming from walks in $\cup_{\gamma': |\gamma'| > 2t/3} W_{\gamma'}$ (i.e. from walks γ 's with less than $t/3$ lazy steps) is negligible (the expected number of walkers who performed in their first t steps a walk with less than $t/3$ lazy steps decays asymptotically faster than any positive power of $1/|V|$, as $|V| \rightarrow \infty$). Moreover, we argue that for every non-lazy walk γ' of length at most $2t/3$ which starts at some vertex belonging to B_i and ends at v , the total contribution to $\mu_v(i)$ coming from walks in $W_{\gamma'}$ (which equals $\sum_{\gamma \in W_{\gamma'}} p(\gamma) X_{\gamma(0)}$), is larger than the contribution to $\mu_u(i)$ coming from walks in $W_{\tilde{\gamma}}$ (which equals $\sum_{\gamma \in W_{\tilde{\gamma}}} p(\gamma) X_{\gamma(0)}$), by a factor of at most d/c_2 , where $\tilde{\gamma}$ is the non-lazy walk obtained from γ' by adding an additional step towards u at the end of γ' (recall the notation from § 2: $p(\gamma)$ is the probability that a walker starting from $\gamma(0)$ would perform the walk γ). We leave the precise details as an exercise and settle with the following sketch of the proof: $1/(2d)$ is (a bound on) the “cost” of making the last step towards u and since the total number of lazy steps performed is at least $t/3$, the “cost” of making one less lazy step is bounded.

Putting the last two observations together, it follows that $\mu_u(i) \geq (c_3/d)\mu_v(i) \geq c_3(1 - 2\delta)\bar{\pi}_v/d$, which implies (3.11), as desired.

We now prove part (iii). We fix some i . The case that $\mu_v(i) \leq (1 - 2\delta)\bar{\pi}_v$ for all v is the same as before. We now treat the case that $\mu_v(i) > (1 - 2\delta)\bar{\pi}_v$ for some v but not for all v . We may assume that $|\mathcal{A}_i| \leq \frac{1}{2}(|V| + |V|^{2/3})$, as there can be at most one class which contains more than half of the walkers. For every $u \in D_i$, let $m_u := |\{v \notin D_i : \{u, v\} \in E\}|$. Let $F_i := \{u \in D_i : m_u > 0\}$. Let Z_u be the indicator of the event that there exists at least one walker from \mathcal{A}_i at u at time $t-1$. By the previous analysis, there exists some $c > 0$ such that $\mathbb{E}[Z_u] > c$ for all $u \in D_i$. By assumptions (a) and (b) from part (iii) this implies that $\mathbb{E}[Z] > c_4 d$, where $\sum_{u \in F_i} m_u Z_u$. It is easy to see that $\text{Cov}(Z_u, Z_v) \leq 0$ for all $u \neq v$. Using the one-sided Chebyshev inequality we get that $\mathbb{P}[Z > c_4 d/2] > c_5$. This clearly implies that the probability that at least one vertex in $V \setminus D_i$ is occupied at time t by at least one walker from \mathcal{A}_i is bounded from below by some positive constant, c_7 . By the definition of D_i , the probability that this occurs and that in one of those vertices there will be at time t also a walker not from \mathcal{A}_i is at least $c_7(1 - e^{-\delta})$. \square

Lemma 3.8. *Let ξ_1, \dots, ξ_m be independent Bernoulli random variables. Denote $p_i := \mathbb{E}[\xi_i]$, $p_* := \max p_i$, $S := \sum_i \xi_i$ and $\mu = \mathbb{E}[S]$. Then $\mathbb{P}[S = 0] = \prod_{i=1}^m (1 - p_i) \leq \exp[\sum -p_i] = e^{-\mu}$.*

Remark 3.9. *The proof above works as long as the holding probability is $\Omega(1/\log |V|)$.*

3.1 Proof of proposition 3.4

Proof: By [6, Lemma 2.4] there exists some $C > 0$ such that for $t := \lceil C|E||V| \rceil$

$$p^t(x, y) \geq \pi_y/2, \quad \text{for all } x, y \in V.$$

Hence for every pair of walkers with some arbitrary initial positions we have that the probability that they are at the same position at time t is at least $\sum_{x \in V} (\pi_x/2)^2 \geq \frac{1}{4|V|} (\sum_x \pi_x)^2 \geq \frac{1}{4|V|}$. We may assume that the total number of walkers Y is greater than $3|V|/4$, as the probability that this fails is exponentially small in $|V|$. Under this assumption, it follows from Lemma 3.2 that there exists some $\alpha > 0$ such that for every collection of $1 + \frac{1}{2}\lceil 3|V|/4 \rceil$ walkers with some arbitrary initial conditions, the probability that the first walker met at time t one of the other walkers in the collection is at least α . It follows that (if $Y > \frac{3|V|}{4}$) the configuration of walkers at each fixed time s , deterministically, has the α -merging property w.r.t. t . \square

4 General lower bounds on SC

Clearly, we can bound SC from below by the minimal time in which every walker has met at least one other walker. This motivates the following definition.

Definition 4.1. *We say that a walker remained **isolated** by time t if this walker did not meet any other walkers by time t (including). We say that a walker is **lazy** by time t if this walker has not left her initial position by time t . We denote the event that there is an isolated lazy walker at vertex v by time t by $\text{IL}_v(t)$.*

Recall the notation from § 2. The following proposition follows from Fact 2.1 in a straightforward manner.

Proposition 4.2. *Let $\gamma = (\gamma(0), \dots, \gamma(t)) \in \Gamma_t$. Then, given that $X_\gamma = 1$, the conditional distribution of the number of walkers that the walker who performed the path γ met by time t is $\text{Poisson}(a_\gamma)$, where $a_\gamma := \sum_{\gamma' \neq \gamma: \exists i, \gamma_i = \gamma'_i} q(\gamma') \leq -q(\gamma) + \sum_{i=0}^t \bar{\pi}_{\gamma_i}$. In particular,*

$$\begin{aligned} \forall v, \quad & \mathbb{P}[\text{IL}_v(t)] \geq 2^{-t} \bar{\pi}_v e^{-(t+1)\bar{\pi}_v}, \\ \forall v \neq u, \quad & \mathbb{P}[\text{IL}_v(t) \cap \text{IL}_u(t)] = \mathbb{P}[\text{IL}_v(t)] \mathbb{P}[\text{IL}_u(t)] e^{m_{u,v}(t)}, \end{aligned} \tag{4.1}$$

where

$$m_{u,v}(t+1) := \sum_{\gamma \in \Gamma_{t+1}: v, u \in \gamma, \gamma_0 \notin \{v, u\}} q(\gamma) \leq t \bar{\pi}_v \mathbb{P}_v[T_u \leq t] + t \bar{\pi}_u \mathbb{P}_u[T_v \leq t] = 2t \bar{\pi}_v \mathbb{P}_v[T_u \leq t].$$

Proof. For the inequality, note that the probability that there is one walker at v at time 0 and that this walker is lazy by time t is $2^{-t} e^{-\bar{\pi}_v}$. The conditional probability that no other walker visited v by time t is $\mathbb{P}[\text{Pois}(\sum_{\gamma \in \Gamma_t: v \in \gamma, \gamma_0 \neq v} q(\gamma)) = 0] \geq \mathbb{P}[\text{Pois}(t \bar{\pi}_v) = 0] \geq e^{-t \bar{\pi}_v}$.

We now prove the second line of (4.1). Let $\Gamma_1 := \{\gamma \in \Gamma_t : v, u \in \gamma, \gamma_0 \notin \{v, u\}\}$, $\Gamma_2 := \{\gamma \in \Gamma_t : v \in \gamma, u \notin \gamma, \gamma_0 \neq v\}$ and $\Gamma_3 := \{\gamma \in \Gamma_t : u \in \gamma, v \notin \gamma, \gamma_0 \neq u\}$. Denote $b_i := \sum_{\gamma \in \Gamma_i} q(\gamma)$ ($i = 1, 2, 3$). Then $b_1 = m_{u,v}(t)$ and by the above reasoning

$$\begin{aligned} P[\text{IL}_v(t)] &= 2^{-t} e^{-\bar{\pi}_v} e^{-(b_1+b_2)}, \quad P[\text{IL}_u(t)] = 2^{-t} e^{-\bar{\pi}_u} e^{-(b_1+b_3)}. \\ P[\text{IL}_v(t) \cap \text{IL}_u(t)] &= (2^{-t} e^{-\bar{\pi}_v})(2^{-t} e^{-\bar{\pi}_u}) e^{-(b_1+b_2+b_3)} = P[\text{IL}_v(t)]P[\text{IL}_u(t)]e^{b_1}. \end{aligned}$$

□

The following theorem implies (1.2).

Theorem 4.3. *Let $G = (V, E)$ be a connected n -vertex graph. Then*

$$P[\cup_v \text{IL}_v(t)] \geq 1 - \exp(-n(\pi_*/\pi^*)^2/[6(2e^2)^{2(t+1)}(t+1)^2]) - e^{-n/3}, \quad \text{for all } t \geq 0, \quad (4.2)$$

where $\pi_* := \min_v \pi_v$ and $\pi^* := \max_v \pi_v$. Hence, if $\pi_*/\pi^* \geq \sqrt{6c_1}n^{\alpha-1/2}$ for some $\alpha, c_1 \in (0, 1]$, then for $t_\alpha := \lfloor c'_\alpha \log n - \log \log n \rfloor - 1$, where $c'_\alpha := \frac{\alpha}{2 \log(2e^2)}$, we have that

$$P[\cup_v \text{IL}_v(t_\alpha)] \geq 1 - \exp[-c_1 n^\alpha] - e^{-n/3}.$$

Proof. Let $A := \{v : \bar{\pi}_v \leq 2\}$. Then $|A| \geq n/2$. Fix $t \geq 0$. Let $Z_v := 1_{\text{IL}_v(t)}$. By (4.1),

$$\sum_{a \in A} \mathbb{E}[Z_a] \geq |A| n \pi_* 2^{-t} e^{-2(t+1)} \geq n^2 \pi_*/(2e^2)^{t+1}. \quad (4.3)$$

We consider the following construction of the SN model. Place at each v an infinite sequence of “candidate walkers” w_v^1, w_v^2, \dots performing independent LSRWs $(\mathbf{w}_v^1(s))_{s \geq 0}, (\mathbf{w}_v^2(s))_{s \geq 0} \dots$. Let $N_v \sim \text{Pois}(\bar{\pi}_v)$ (independent for different v ’s and independent of the walks). Finally, censor the walks performed by walkers of the form w_v^i such that $i > N_v$. Consider the Doob’s sequence of $Z := \sum_{a \in A} Z_a$ w.r.t. the following exposure procedure (i.e. the martingale $M_k := \mathbb{E}[Z \mid \mathcal{F}_k]$, where \mathcal{F}_k is the natural filtration of the exposure procedure). We label the vertices as v_1, \dots, v_n . We expose the walkers and their walks one at a time as follows. Starting from $i = k = 1$ we expose in each round a pair of the form $(1_{N_{v_i} \geq k}, \mathbf{w}_{v_i}^k)$, where whenever we expose a pair of the form $(1, \mathbf{w}_{v_i}^k)$ (resp. $(0, \mathbf{w}_{v_i}^k)$), the next pair to be expose is of the form $(1_{N_{v_i} \geq k+1}, \mathbf{w}_{v_i}^{k+1})$ (resp. $(1_{N_{v_{i+1}} \geq 1}, \mathbf{w}_{v_{i+1}}^1)$). The exposure procedure has more than $3n$ steps only if $\sum_v N_v > 2n$, which by (2.2) has probability of at most $e^{-n/3}$. Thus by (4.3) and Azuma inequality it suffices to show that the absolute values of the increments of the corresponding Doob’s martingale are bounded by $n\pi^*(t+1)$.

Indeed, in a step in which we exposed $(1_{N_{v_i} \geq k}, \mathbf{w}_{v_i}^k)$ our current “best estimate” on N_{v_i} changed (in absolute value) by at most $\mathbb{E}[N_{v_i}] \leq n\pi^*$ while the possible contribution in absolute value to Z of each walker is at most $t+1$. The last two claims are left as an exercise. The former can be proven by showing that for every $\lambda > 0$ and $k \geq 0$ if μ_k is the $\text{Pois}(\lambda)$ distribution, conditioned to be at least k , then μ_{k+1} is stochastically dominated by $\nu_k(i) := \mu_k(i-1)$ (i.e. by “ $1 + \mu_k$ ”, in particular, we have that “ $\mu_k - k$ ” is stochastically dominated by the $\text{Pois}(\lambda)$ distribution). So in each step our current estimate for the expected number of walkers at a certain site can increase by at most 1 and decrease by at most $\mathbb{E}[\text{Pois}(n\pi^*)] = n\pi^*$. For the latter claim, we note that the walk of each candidate walker by time t can be “pivotal” for at most $t+1$ Z_a ’s, regardless of the walks performed by all of the remaining candidate walkers whose walk by time t is still unexposed. □

Theorem 4.4. Let $\mu := \sum_{v: \bar{\pi}_v \leq 2} \bar{\pi}_v$ and $a_t := \mu 2^{-t} e^{-2(t+1)}$. Then for all $t \geq 0$,

$$\mathbb{P}[\cup_v \text{IL}_v(t)] \geq 1 - 7t/\sqrt{a_t}. \quad (4.4)$$

Hence, if there exists some $\delta > 0$ such that $\mu \geq c_1 n^\beta$ for some $\beta > 0$, then for $s_\beta := \max(\lfloor \tilde{c}_\beta \log n - 2 \log \log n \rfloor - 1, 0)$, where $\tilde{c}_\beta := \frac{\beta}{2 \log(2e^2)}$, we have that

$$\mathbb{P}[\cup_v \text{IL}_v(s_\beta)] \geq 1 - 12n^{-\beta/4}.$$

Proof. We may assume that $7t \leq \sqrt{a_t}$. Let $A := \{v : \bar{\pi}_v \leq 2\}$. Consider

$$B_a := \{b \in A : \bar{\pi}_a \mathbb{P}_a[T_b \leq t] \geq \bar{\pi}_b/(2\sqrt{a_t})\}.$$

For every $a \in A$, $\sum_{b \in A} \mathbb{P}_a[T_b \leq t] \leq t$ and so $\sum_{b \in B_a} \bar{\pi}_b \leq 2\bar{\pi}_a \sqrt{a_t} \sum_{b \in B_a} \mathbb{P}_a[T_b \leq t] \leq 2\bar{\pi}_a t \sqrt{a_t}$. This (together with $7t \leq \sqrt{a_t}$, which implies that $t\sqrt{a_t} > 1$) implies that there exists some $D \subset A$ such that

$$\sum_{a \in D} \bar{\pi}_a \geq \mu/(1 + 2t\sqrt{a_t}) \geq \mu/(3t\sqrt{a_t}). \quad (4.5)$$

$$\bar{\pi}_a \mathbb{P}_a[T_b \leq t] < \max(\bar{\pi}_a, \bar{\pi}_b)/(2\sqrt{a_t}), \text{ for all } a, b \in D. \quad (4.6)$$

By (4.1), $\mathbb{E}[Z] \geq \sqrt{a_t}/(3t)$, where $Z := \sum_{a \in D} Z_a$ and $Z_a := 1_{\text{IL}_a(t)}$. For all $a, b \in D$, $2\bar{\pi}_a t \mathbb{P}_a[T_b \leq t] \leq 4t/\sqrt{a_t} \leq 4/7$. Thus

$$\exp(2\bar{\pi}_a t \mathbb{P}_a[T_b \leq t]) - 1 \leq 4\bar{\pi}_a t \mathbb{P}_a[T_b \leq t] \leq 2 \max(\bar{\pi}_a, \bar{\pi}_b) t / \sqrt{a_t} \leq 4t/\sqrt{a_t},$$

and so by (4.1)

$$\text{Cov}(Z_a, Z_b)/(\mathbb{E}[Z_a]\mathbb{E}[Z_b]) \leq \exp(2\bar{\pi}_a t \mathbb{P}_a[T_b \leq t]) - 1 \leq 4t/\sqrt{a_t}.$$

Hence $L := \sum_{a \neq b, a, b \in D} \text{Cov}(Z_a, Z_b) \leq (\mathbb{E}[Z])^2 (4t/\sqrt{a_t})$. Finally, since $\text{Var} Z = \mathbb{E}[Z] + L$ by Chebyshev's inequality we have that $\mathbb{P}[Z = 0] \leq \frac{\text{Var} Z}{(\mathbb{E}[Z])^2} \leq \frac{1}{\mathbb{E}[Z]} + 4t/\sqrt{a_t} \leq 7t/\sqrt{a_t}$. \square

The next example demonstrates that the assertion of Theorem 4.4 is quite sharp.

Example 4.5. Denote $L_n := \lceil \log^{10} n \rceil$. Consider a $\lceil n/L_n \rceil$ -clique, such that each vertex in the clique is the center of a star of size L_n (where all stars are disjoint). Then one can show that $\text{SC} \leq C \log \log n$ w.h.p. (and if the walkers perform non-lazy SRWs we would get that $\text{SC} \leq C$ w.h.p.). The reason for this is that the expected number of walkers whose initial position is not in a center of a star is at most L_n^2 , while at each vertex of the clique there are $\text{Pois}(\mu_n)$ walkers for some $\mu_n \approx L_n$. An easy calculation shows that w.h.p. after one step all of the walkers whose initial positions are in the clique are in the same class.

Proof of Theorem 2: Let $G = (V, E)$ be a connected n -vertex d -regular graph. Consider the SN model on G in which the walks have some fixed holding probability $p \in [0, 1)$. Denote its transition matrix by P . Recall that Γ_t is the collection of all walks on G of length t and that for $\gamma = (\gamma_0, \dots, \gamma_t) \in \Gamma_t$, $p(\gamma) = \prod_{i=0}^{t-1} p^i(\gamma_i, \gamma_{i+1})$. Fix $t = t_n = \lfloor \log n - 6 \log \log n \rfloor - 1$.

Assume $t \geq 0$ as otherwise there is nothing to prove. For every $\Gamma' \subset \Gamma_t$ let $m(\Gamma') := \sum_{\gamma \in \Gamma'} p(\gamma)$. For every $\gamma \in \Gamma_t$ and $0 \leq i \leq t$ let

$$B_{\gamma,i} := \{v \in V : \max_{j: 0 \leq j \leq t} p^{|i-j|}(\gamma_j, v) \geq (t+1)^{-3}\},$$

$$\text{Bad}_\gamma := \{\tilde{\gamma} \in \Gamma_t : \exists i \in \{0, 1, \dots, t\} \text{ such that } \tilde{\gamma}_i \in B_{\gamma,i}\}.$$

By Markov inequality $|B_{\gamma,i}| \leq (t+1)^4$, for all $\gamma \in \Gamma_t$ and $i \in \{0, 1, \dots, t\}$, and thus

$$m(\text{Bad}_\gamma) \leq \sum_{i \leq t} \sum_{v \in B_{\gamma,i}} m(\{\tilde{\gamma} \in \Gamma_t : \tilde{\gamma}_i = v\}) \leq \sum_{i \leq t} |B_{\gamma,i}| \leq (t+1)^5. \quad (4.7)$$

Fix some order on Γ_t . Recall that X_γ denotes the number of walkers who performed the walk γ . Expose X_γ for some $\gamma \in \Gamma_t$ according to the following procedure. Assuming that we have already exposed $A \subset \Gamma_t$ so that the following holds: for every $\gamma \in A$ with $X_\gamma > 0$ we have that $\text{Bad}_\gamma \cap A = \{\gamma\}$. In the next stage we expose $X_{\gamma'}$ for the minimal $\gamma' \in \Gamma_t \setminus B(A)$, where $B(A) := \cup_{\gamma \in A: X_\gamma > 0} \text{Bad}_\gamma$. In the following stage we apply the same rule, with the set A replaced by the set $A \cup \{\gamma'\}$. At the end of this procedure we obtain a collection $\mathcal{W} = \{\gamma^1, \dots, \gamma^{|\mathcal{W}|}\}$ (resp. \mathcal{N}) of all γ 's for which we exposed that $X_\gamma > 0$ (resp. $= 0$), where the indices are taken so that $i < j$ iff γ^i is before γ^j in the ordering. The collection of all $\gamma \in \Gamma_t$ for which X_γ was not exposed is precisely $\cup_{\gamma \in \mathcal{W}: X_\gamma > 0} \text{Bad}_\gamma \setminus \{\gamma\} = \Gamma_t \setminus (\mathcal{W} \cup \mathcal{N})$.

Let \mathcal{N}_i be the collection of all $\gamma \in \mathcal{N}$ which were exposed in between γ^{i-1} and γ^i (for $i = 1$, all γ 's exposed prior to γ^1). For every $1 \leq i \leq |\mathcal{W}|$, the probability that $m(\mathcal{N}_i) > (t+1)^5$ is at most $\mathbb{P}[\text{Pois}((t+1)^5) = 0] = e^{-(t+1)^5}$. Hence by (4.7)

$$\mathbb{P}[|\mathcal{W}| < n/(2(t+1)^5)] \leq ne^{-(t+1)^5} = o(1). \quad (4.8)$$

We now condition on $\mathcal{W} = W$ and $\mathcal{N} = N$ for some arbitrary $W, N \subset \Gamma_t$ so that (i) $|\mathcal{W}| \geq n/[2(t+1)^5]$ and (ii) $\text{Bad}_\gamma \cap W = \{\gamma\}$, for all $\gamma \in W$. Observe that, given $(\mathcal{W}, \mathcal{N}) = (W, N)$, the joint distribution of $(X_\gamma)_{\gamma \in \Gamma_t \setminus (W \cup N)}$ is the same as their unconditional joint distribution. For $\gamma = (\gamma_0, \dots, \gamma_t)$, $\gamma' = (\gamma'_0, \dots, \gamma'_t) \in W$ let

$$\Gamma_\gamma := \{\tilde{\gamma} \in \Gamma_t \setminus (N \cup \{\gamma\}) : \exists i \in \{0, 1, \dots, t\} \text{ such that } \tilde{\gamma}_i = \gamma_i\} \subset \text{Bad}_\gamma,$$

$$\Gamma_{\gamma, \gamma'} := \{\tilde{\gamma} \in \Gamma_t : \exists i, j \in \{0, 1, \dots, t\} \text{ such that } \tilde{\gamma}_i = \gamma_i \text{ and } \tilde{\gamma}_j = \gamma'_j\} \subset \text{Bad}_\gamma \cap \text{Bad}_{\gamma'}.$$

Note that $m_\gamma := m(\Gamma_\gamma) \leq \sum_{i=0}^t p(\{\tilde{\gamma} \in \Gamma_t : \tilde{\gamma}_i = \gamma_i\}) \leq t+1$. Denote

$$D_{i,j}(\gamma, \gamma') := \{\tilde{\gamma} \in \Gamma_t : \tilde{\gamma}_i = \gamma_i, \tilde{\gamma}_j = \gamma'_j, \forall k < i \tilde{\gamma}_k \neq \gamma_k, \forall r < j \tilde{\gamma}_r \neq \gamma'_r\}$$

By construction $\gamma' \notin \text{Bad}_\gamma$, for all $\gamma, \gamma' \in W$ and so $\max_{i,j: 0 \leq i,j \leq t} p^{|i-j|}(\gamma_i, \gamma'_j) < (t+1)^{-3}$. Hence by reversibility

$$m_{\gamma, \gamma'} := m(\Gamma_{\gamma, \gamma'}) = \sum_{0 \leq i < j \leq t} m(D_{i,j}(\gamma, \gamma')) + m(D_{i,j}(\gamma', \gamma)) \leq \sum_{0 \leq i, j \leq t} p^{|i-j|}(\gamma_i, \gamma'_j) \leq (t+1)^{-1}.$$

Finally, for each $\gamma \in W$ set Z_γ to be the indicator of the event that $\sum_{\gamma \in \Gamma_\gamma} X_\gamma = 0$. Set $Z = \sum_{\gamma \in W} Z_\gamma$. Observe that $\mathbb{E}[Z] \geq |W|e^{-(t+1)} > n/[2(t+1)^5](n^{-1} \log^6 n) > \log n$ (using $m_\gamma \leq t+1$ for all $\gamma \in W$) while (similarly to (4.1)) for all $\gamma, \gamma' \in W$

$$\text{Cov}(Z_\gamma, Z_{\gamma'})/(\mathbb{E}[Z_\gamma]\mathbb{E}[Z_{\gamma'}]) = (e^{m_{\gamma, \gamma'}} - 1) \leq (e^{(t+1)^{-1}} - 1) =: \delta_n,$$

and thus

$$\text{Var}(Z) \leq \mathbb{E}[Z] + \delta_n(\mathbb{E}[Z])^2,$$

which by Chebyshev implies that $\mathbb{P}[Z = 0] < \frac{1}{\mathbb{E}[Z]} + \delta_n$. \square

5 The cycle

In this section we consider the case that G is the n -cycle, C_n . In this section we assume that at time 0 there is precisely one walker at each site and that the walkers perform independent continuous-time random walks with jump rate 1.

We first prove (1.3). We fix $s = s_n := C_1 \log^2 n$. By a union bound over the n edges of the cycle it suffices to show that for every pair of neighboring vertices u, v , the probability that the walkers who started at those vertices do not have a path of acquaintances by time s is at most Cn^{-2} , in order to deduce that $\mathbb{P}[\text{SC}(C_n) \geq s] \leq C/n$.

Consider some fixed edge $e = \{u, v\}$. It partitions the vertices into 2 sides (segments) of size roughly $n/2$, A_u, A_v , according to the identity of the end-point to which they are closer to (break a tie arbitrarily). Let e' be the other edge, apart from e , which connects A_u and A_v . Let $\text{CROSS}_s(e, w)$ be the event that the walker who started at vertex w did not cross e' by time s and that her position at time s is at the opposite side of e compared to w .

We now argue that for every $(w, w') \in A_u \times A_v$ the event $\text{CROSS}_s(e, w) \cap \text{CROSS}_s(e, w')$ “forces” the walkers who started at u and v to have a path of acquaintances by time s (which uses only walkers whose starting positions are in $\{u, v, w, w'\}$, possibly a subset of this set). Indeed, the continuous-time setup eliminates the possibility of two walkers swapping positions without meeting. This implies that on the event $\text{CROSS}_s(e, w) \cap \text{CROSS}_s(e, w')$ it must be the case that by time s the walkers whose starting positions are w and w' , resp., have met, and that each of the walkers whose starting positions are u and v , resp., must have met at least one of the previous pair of walkers (in fact, this is the case for any pair of vertices u', v' lying in the segment between w and w' which contains e).

This reduces the proof of (1.3) into the following simple calculation: If C_1 is taken to be sufficiently large then $\mathbb{P}[\text{CROSS}_s(e, w)] \geq 0.4$ for every edge $e = \{u, v\}$ and vertex w of distance at most $10 \log n$ from e (for all sufficiently large n so that the probability that the walker who started at such w crossed e' is at most, say $1/20$), and so (using $(\frac{13}{20})^{10 \log n} \leq n^{-2}$)

$$\mathbb{P}[\text{CROSS}_s(e)] \geq 1 - Cn^{-2}, \text{ where } \text{CROSS}_s(e) := \bigcup_{w \in A_u, w' \in A_v} \text{CROSS}_s(e, w) \cap \text{CROSS}_s(e, w').$$

This concludes the proof of (1.3). We now prove (1.4). We fix $t = t_n = c_1 \log^2 n$ for some constant c_1 to be determined later. Observe that if there are at least 2 edges which no walker crossed by time t , then deterministically, $\text{SC}(C_n) > t$. Let $\text{NC}_t(e)$ be the event that

the edge $e \in E(C_n)$ was not crossed by any walker by time t . We argue that if c_1 is taken to be sufficiently small, then for all sufficiently large n and every edge e it holds that

$$\mathbb{P}[\text{NC}_t(e) \mid F] \geq c_3 n^{-1/4}, \quad (5.1)$$

where F is the event that no walker performed at least $(\sqrt{n}/2) - 2$ steps by time t . We now explain how (1.4) can be derived from (5.1). Denote $m := \lfloor \sqrt{n} \rfloor$. Let $(e_i)_{i=1}^m$ be a collection of edges which are all at distance at least \sqrt{n} from one another. Then (by a union bound over the n walkers) $1 - \mathbb{P}[F] \leq C_2 n e^{-cn^{1/4}} \leq C_2' e^{-c'n^{1/5}}$. Finally, note that conditioned on F , the events $(\text{NC}_t(e_i))_{i=1}^m$ become independent, and so the conditional probability that none of them occur is at most $(1 - c_3 n^{-1/4})^m \leq e^{-c_3' n^{1/4}}$.

We now prove (5.1). It is not hard to see that it suffices to show that for every k if we put one particle in each site of $\mathbb{N} = \{1, 2, \dots\}$ and the particles perform independent continuous-time SRW on \mathbb{Z} with jump rate 1, then the probability that no walker reached the origin by time k is at least $e^{-M\sqrt{k}}$, for some absolute constant M . Denote the probability of the previous event by p_k . Let $(S_r^i)_{r \geq 0}$ be a SRW on \mathbb{Z} , starting at $i \in \mathbb{N}$. Then by the reflection principle (see e.g. [14, § 2.7]; the proof in continuous-time is analogous)

$$a_i(k) := \mathbb{P}[S_r^i > 0, \forall r \in [0, k]] = \mathbb{P}[S_k^0 \in \{-i+1, \dots, i\}], \text{ for all } i \in \mathbb{N}.$$

Write $b := \lfloor 4\sqrt{k} \rfloor$. We argue that there exist constants $c_5, c_6, C_3 > 0$ such that

$$a_i(k) \geq c_5 i/b, \quad \text{for all } 1 \leq i \leq b. \quad (5.2)$$

$$\begin{aligned} a_i(k) &\geq 1 - 2 \exp[-(i^2/2k) + i^4/4k^3] \geq 1 - 2e^{-i^2/(8k)} \geq e^{-C_3 e^{-i^2/(8k)}}, \text{ for } b < i \leq 1.2k. \\ a_i(k) &\geq 1 - e^{-c_6 i} \geq e^{-C_3 e^{-c_6 i}}, \quad \text{for all } i > 1.2k. \end{aligned} \quad (5.3)$$

The first line follows from the local CLT. The first inequality in the third line follows from the fact that $1 - a_i(k) \leq \mathbb{P}[\text{Pois}(k) \geq i]$. The last inequalities in the second and third lines follows from the fact that for every $c > 1$ there exists $C > 0$ such that $1 - x \geq e^{-Cx}$, for all $x \geq c$. The first inequality in the second line is obtained by noting that for discrete-time SRW starting at the origin, $(\tilde{S}_r)_{r \in \mathbb{Z}_+}$, we have that $\mathbb{E}[e^{\lambda \tilde{S}_r}] = (\frac{1}{2}e^\lambda + \frac{1}{2}e^{-\lambda})^r \leq e^{\lambda^2 r/2}$ (by comparing Taylor expansion coefficients). Hence,

$$\mathbb{E}[e^{\lambda S_k}] \leq \sum_r \mathbb{P}[\text{Pois}(k) = r] e^{\lambda^2 r/2} = \exp[k(e^{\lambda^2/2} - 1)] \leq \exp[k(\lambda^2/2 + (\lambda^2/2)^2)],$$

as long as $\lambda^2/2 \leq 1$ (using $e^b - 1 \leq b + b^2$ for all $b \in [-1, 1]$). We set $\lambda = i/k$, so that indeed $\lambda^2/2 \leq 1$ for $i \leq 1.2k$. Finally, by Markov inequality and our choice of λ

$$1 - a_i(k) \leq 2\mathbb{P}[S_k \geq i] \leq 2\mathbb{E}[e^{\lambda S_k}] e^{-\lambda i} \leq 2 \exp[-(i^2/2k) + i^4/4k^3].$$

We are now in a position to conclude the proof. By (5.2) and Stirling's approximation

$$\prod_{i \leq b} a_i(k) \geq c_5^b b! / b^b \geq \sqrt{b} (c_5/e)^b \geq e^{-C_4 \sqrt{k}}.$$

Denote $b' := \lfloor \sqrt{k} \rfloor$. Finally, using (5.3) it is not hard to show that

$$\prod_{i>b} a_i(k) \geq \prod_{\ell: 4 \leq \ell \leq 1.2b'} a_{\ell b'}^{b'}(k) \prod_{\ell > 1.2b'} a_\ell(k) \geq e^{-C_5 \sqrt{k}}$$

(we leave the details to the reader; Alternatively, since for $i > b$, $a_i(k)$ are uniformly bounded away from 0, $\prod_{i>b} a_i(k) \geq e^{-C \sum_{i>b} (1-a_i(k))}$, and the reasoning in the following remark yields that $\sum_{i>0} (1-a_i(k)) = \sum_{i>0} \mathbb{P}_i[T_0 \leq k] \leq C' \sqrt{k}$). We are done as $p_k := \prod_{i>0} a_i(k)$. \square

Remark 5.1. *Consider the usual Poisson setup. The proof of the upper bound on $\text{SC}(C_n)$ is essentially identical to the case of one walker per site at time 0. The proof of the lower bound on $\text{SC}(C_n)$ however becomes much simpler. For simplicity, we explain this in the discrete-time setup. Denote $\nu_t := \sum_{i=0}^t p^i(v, v) = \Theta(\sqrt{t+1})$. By Fact 2.1, the total number of walkers to reach vertex v by time t has a Poisson distribution with mean (by reversibility)*

$$\mu_t := \sum_u \mathbb{P}_u[T_v \leq t] \leq \frac{1}{\nu_t} \sum_u \sum_{j=0}^t \mathbb{P}_u[T_v = j] \sum_{i=0}^{2t-j} p^i(v, v) \leq \frac{C}{\sqrt{t+1}} \sum_u \sum_{i=0}^{2t} p^i(u, v) \leq \frac{C(2t+1)}{\sqrt{t+1}}.$$

Thus if $t+1 < (\frac{\log n}{8C})^2$ we get that the probability that no walker reached v by time t is at least $e^{-\mu_t} \geq n^{-1/4}$. This implies the lower bound on $\text{SC}(C_n)$ as in the proof above.

6 Expanders

In this section we study the case that G is a d -regular expander. It is not difficult to extend the results to the case G is an expander of maximal degree d . LSRW on a regular λ -expander $G = (V, E)$ mixes rapidly in the following sense

$$\max_{x,y \in V} |p^t(x, y) - \pi_y| \leq (1 - \lambda)^t, \quad \text{for all } t. \quad (6.1)$$

Indeed $\max_{x \in V} |p^t(x, x) - \pi_x| \leq (1 - \lambda)^t$ follows from the spectral decomposition of $p^t(x, x)$ along with the non-negativity of the eigenvalues of P . However, for LSRW on a regular graph $\max_{x,y \in V} |p^t(x, y) - \pi_y| = \max_{x \in V} |p^t(x, x) - \pi_x|$, since in general (cf. [7, (2.2)] for even t and [14, p. 135] for moving from even t to odd t using laziness)

$$\max_{x,y} |p^t(x, y) - \pi_y| \leq \sqrt{\max_{u,v} (\pi_u / \pi_v)} \max_x |p^t(x, x) - \pi_x|.$$

6.1 Proof of Theorem 5

We argue that Theorem 5 follows from Theorem 6. This follows at once from Corollary 6.2. Indeed, after a (“giant”) class of walkers, \mathcal{A} , of size at least $n/6$ emerges, by Corollary 6.2 (using a union bound over the walkers not in \mathcal{A} , together with the concentration of the total number of walkers around n), w.p. at least $1 - C_2 n^{-1}$, the additional amount of time until every walker not from \mathcal{A} will meet some walker from \mathcal{A} is at most $\lceil C \lambda^{-1} \log n \rceil$. \square

Lemma 6.1. *Let G be a connected d -regular n -vertex λ -expander. Then there exists $C > 0$ such that for $t := \lceil C \lambda^{-1} \log n \rceil$ and every v, u_1, u_2, \dots, u_t sequence of vertices, the probability that a LSRW started at v visits some u_i at time i for some $1 \leq i \leq t$ is at least $\frac{1}{8} \min(\frac{\lambda t}{n}, 1)$.*

Proof. Let $(X_s)_{s \geq 0}$ be a LSRW on G started at v . Let $Y_i = 1_{X_i = u_i}$ and $Y := \sum_{i=1}^t Y_i$, where $t := \lceil C\lambda^{-1} \log n \rceil$, for some constant C , to be determined shortly. By (6.1), if C is sufficiently large, $\mathbb{E}[Y] \geq t/(2n)$, whereas $\mathbb{E}[Y_i Y_{i+j}] \leq \mathbb{E}[Y_i] p^j(u_i, u_{i+j}) \leq \mathbb{E}[Y_i] (n^{-1} + (1 - \lambda)^j)$ and so $\mathbb{E}[Y^2] \leq 2\mathbb{E}[Y](\lambda^{-1} + t/n)$. Finally, $\mathbb{P}[Y > 0] \geq \frac{(\mathbb{E}[Y])^2}{\mathbb{E}[Y^2]} \geq \frac{\mathbb{E}[Y]}{2(\gamma^{-1} + \frac{t}{n})} \geq \frac{1}{8} \min(\frac{\lambda t}{n}, 1)$. \square

Corollary 6.2. *Let $G = (V, E)$ be a connected d -regular n -vertex λ -expander. Let $\ell := \lceil n/6 \rceil$. Let $(v_i)_{i=0}^\ell$ be an arbitrary sequence of vertices. Denote $a_v := |\{0 \leq i \leq \ell : v_i = v\}|$. Assume that at time 0 there are a_v particles at vertex v for all $v \in V$ and that the particles perform independent LSRWs on G . Assume that $a_{v_0} = 1$. Let A_s be the event that the particle that started at v_0 met (i.e. visited the same vertex at the same time) at least one other particle by time s . Then there exist some constants C, n' (independent of G, d, λ and the sequence $(v_i)_{i=0}^\ell$) such that $\mathbb{P}[A_t] \geq 1 - n^{-2}$, for $t := \lceil C\lambda^{-1} \log n \rceil$, as long as $n \geq n'$.*

Proof. First, condition on the walk that the particle started at v_0 performed by time t , denoted by γ . Then use Lemma 6.1 and independence to obtain the desired estimate for the conditional probability (uniformly, for all possible values of γ). Finally, average over γ . \square

6.2 Proof of Theorem 6

Proof: We adapt a technique from [2, Proposition 3.1] to our setup. Let $t \leq n$ to be determined later. Set $s := \lceil 8(t+2)/\lambda \rceil$. Call a class of walkers at time s **good**, if the collection of walkers from this class occupy at time s at least t vertices. By Corollary 6.4 (if t is sufficiently large), w.p. at least $1 - \exp(-c_1 n/d^{4(s+1)})$, there exists a collection of at most $n/(2t)$ good classes at time s , such that the walkers belonging to the union of these classes occupy at least $n/2$ vertices at time s . We conditioned on this event and on the identity of the good classes from this collection and on the positions at time s of the corresponding walkers. We claim, that if t is taken to be sufficiently large, then w.p. at least $1 - \exp(-c_2 n)$, there is no way of splitting this collection into 2 (disjoint) collections, \mathcal{A}, \mathcal{B} , such that

- (i) The walkers belonging to the union of the classes in \mathcal{A} (resp. \mathcal{B}), occupy at time s at least $n/6$ vertices.
- (ii) No walker from (the union of the classes in) \mathcal{A} met a walker from \mathcal{B} by time $s + r$, where $r := \lceil C\lambda^{-1} \rceil$, for some sufficiently large constant C .

Indeed, this follows from Lemma 6.5, which asserts that for any choice of \mathcal{A}, \mathcal{B} satisfying (i), the probability that (ii) holds is at most $e^{-c_2 n}$, by a union bound over all $\leq 2^{n/(2t)}$ such partitions (and setting $t \geq \lceil 1/c_2 \rceil$). \square

Lemma 6.3. *Let $G = (V, E)$ be a connected d -regular n -vertex λ -expander. Consider the SN model on G . Let $\mathcal{W}_u(s)$ denote the walkers who occupy vertex u at time s . Write $u \leftrightarrow_s v$ if there exists a pair of walkers $(w, w') \in \mathcal{W}_u(s) \times \mathcal{W}_v(s)$ who met each other by time s . Let $A_{u,r}(s)$ be the event that $|\{v : u \leftrightarrow_s v\}| \geq r$. Denote $s := \lceil 8(t+2)/\lambda \rceil$. Then there exists an absolute constant $c > 0$ such that if $s \leq n$ then*

$$\mathbb{P}(A_{u,t}(s)) \geq (1 - e^{-1}) - e^{-ct}.$$

Proof. We have that $P[|\mathcal{W}_u(s)| > 0] = 1 - e^{-1}$. Throughout the proof we condition on $|\mathcal{W}_u(s)| > 0$. Moreover, we fix some walker $w \in \mathcal{W}_u(s)$ and condition on the walk $\mathbf{w} = (\mathbf{w}(0), \dots, \mathbf{w}(s) = u)$ that she performed by time s . By averaging, it suffices to show that the conditional probability that $A_{u,t}(s)$ fails is at most e^{-ct} for some constant $c > 0$ independent of \mathbf{w} . Below, all probabilities and expectations are the conditional ones, given \mathbf{w} .

Let $(X_k)_{k \geq 0}$ be a LSRW on G . Let $\tau_{\mathbf{w}} := \inf\{k : X_k = \mathbf{w}(s - k), k \leq s\}$. By Fact 2.1 and reversibility, the (conditional) distribution (given \mathbf{w}) of the number of walkers not from $\mathcal{W}_u(s)$ that w met by time s has a Poisson distribution with mean $\mu_{\mathbf{w}} := \sum_{v:v \neq u} \mathbb{P}_v[\tau_{\mathbf{w}} \leq s]$. We argue that $\mu_{\mathbf{w}} \geq 4t$. Indeed, let $b_{v,\mathbf{w}} := \sum_{i=0}^s p^i(v, \mathbf{w}(s-i))$. By reversibility and (6.1) $\mu'_{\mathbf{w}} := \sum_{v:v \neq u} b_{v,\mathbf{w}} = s - b_{u,\mathbf{w}} \geq s - \lambda^{-1} - 1$. Again by (6.1) and reversibility, for all v

$$\begin{aligned} b_{v,\mathbf{w}} &= \sum_{i \leq s} \mathbb{P}_v[\tau_{\mathbf{w}} = i] \sum_{j=0}^{s-i} p^j(\mathbf{w}(s-i), \mathbf{w}(s-i-j)) \leq \sum_{j=0}^{s-1} \max_{x,y} p^j(x,y) \mathbb{P}_v[\tau_{\mathbf{w}} \leq s] \\ &\leq \mathbb{P}_v[\tau_{\mathbf{w}} \leq s](\lambda^{-1} + 1) \leq 2\lambda^{-1} \mathbb{P}_v[\tau_{\mathbf{w}} \leq s]. \end{aligned}$$

Thus, by summing over all $v \in V \setminus \{u\}$, $\mu_{\mathbf{w}} \geq (\lambda/2)\mu'_{\mathbf{w}} \geq (\lambda/2)(s - \lambda^{-1} - 1) \geq 4t$, as desired.

Let Z_v be the number of walkers in $\mathcal{W}_v(s)$ who met w by time s . Then (conditioned on \mathbf{w}) $(Z_v)_{v:v \neq u}$ are independent Poisson random variables such that $\mathbb{E}[Z_v] \leq 1$ for all v and $\sum_{v:v \neq u} \mathbb{E}[Z_v] \geq 4t$. Finally, by Lemma 6.6 $\mathbb{E}[1_{Z_v \geq 1}] \geq (1 - e^{-1})\mathbb{E}[Z_v]$ for all v , so $\sum_{v:v \neq u} \mathbb{E}[1_{Z_v \geq 1}] > (1 - e^{-1})4t > 2t$. Thus by Bernstein inequality $P[\sum_{v:v \neq u} 1_{Z_v \geq 0} \leq t] \leq e^{-ct}$, for some constant $c > 0$, as desired. \square

Corollary 6.4. *In the setup and notation of Lemma 6.3, there exist some t_0, c_1 such that*

$$P\left[\sum_{u \in V} 1_{A_{u,t}(s)} < n/2\right] \leq \exp(-c_1 n/d^{4(s+1)}), \text{ for all } t \geq t_0 \text{ such that } s = \lceil 8(t+2)/\lambda \rceil \leq n.$$

Proof. Use Lemma 6.3 together with Azuma inequality on an appropriate Doob's sequence, by exposing the value of indicators on the l.h.s. sequentially, one at a time. The absolute value of the increments are bounded by the maximal size of a ball of radius $2s$ ($\leq d^{2(s+1)}$). \square

Lemma 6.5. *Let $G = (V, E)$ be a connected d -regular n -vertex λ -expander. Let $0 < \delta \leq 1/2$. Let $A, B \subset V$ be two disjoint sets of size at least δn . Put one walker at each vertex of $A \cup B$ and let them perform independent LSRWs. There exists an absolute constant $c > 0$ such that the probability that no walker from A met a walker from B by time $r := \lceil \lambda^{-1} |\log(\delta^2/4)| \rceil$ is at most $\exp[-c\delta^2 n]$.*

Proof. Let $\mathcal{W}_A, \mathcal{W}_B$ be the collection of walkers occupying A and B , respectively, at time 0 (as indicated in the assertion of the lemma, at each vertex there is exactly one walker at time 0). Let D_t be the set of vertices which are occupied at time t by at least one walker in \mathcal{W}_B . Using Lemma 6.6, it is easy to verify that $\mathbb{E}[|D_t|] \geq |B|(1 - e^{-1})$, for every t . Thus by applying Azuma inequality to an appropriate Doob's sequence (by exposing the positions at time t of the walkers in \mathcal{W}_B sequentially, one at a time),

$$P[|D_t| < \delta n/4] \leq e^{-c_1 \delta n}. \quad (6.2)$$

Denote the uniform distribution on A by π_A and by $P_{\pi_A}^r$ the distribution of LSRW with $X_0 \sim \pi_A$. Given that $D_r = D$, the conditional probability that no walker from \mathcal{W}_A met a walker from \mathcal{W}_B at time r is at most

$$\prod_{a \in A} (1 - P^r(a, D)) \leq \exp\left[-\sum_{a \in A} P^r(a, D)\right] = \exp[-|A|P_{\pi_A}^r(D)].$$

To conclude the proof, we show that $P_{\pi_A}^r(D) \geq \delta/8$ for all $D \subset V$ such that $|D| \geq \delta n/4$, which implies the assertion of the Lemma using (6.2).

Note that the ℓ_2 distance of π_A from π (the uniform distribution on V), $\|\pi_A - \pi\|_{2,\pi} := [\sum_{v \in V} (\frac{\pi_A(v)}{\pi(v)} - 1)^2]^{1/2}$, is at most their ℓ_∞ distance, $\|\pi_A - \pi\|_{\infty,\pi} := \max_v |\frac{\pi_A(v)}{\pi(v)} - 1| \leq n/|A|$, (in fact, $\|\pi_A - \pi\|_{2,\pi} = \sqrt{(n - |A|)/|A|}$) and so by Jensen's inequality (first inequality) $\|P_{\pi_A}^r - \pi\|_{1,\pi} \leq \|P_{\pi_A}^r - \pi\|_{2,\pi} \leq (1 - \lambda)^r \|\pi_A - \pi\|_{2,\pi} \leq \frac{n}{|A|} e^{-\lambda r} \leq \delta/4$, by the general Poincaré inequality satisfied by the spectral gap (cf. [1, Lemma 3.26]), together with our choice of r (where $\|P_{\pi_A}^r - \pi\|_{1,\pi} = \sum_v |P_{\pi_A}^r(v) - \pi(v)| = 2 \max_{D \subset V} \pi(D) - P_{\pi_A}^r(D)$). Hence if $|D| \geq \delta n/4$, then $P_{\pi_A}^r(D) \geq \pi(D) - \frac{1}{2} \|P_{\pi_A}^r - \pi\|_{1,\pi} = (|D|/n) - \delta/8 \geq \delta/8$. \square

Lemma 6.6. *For all $x \in [0, 1]$, $(1 - e^{-x}) \geq (1 - e^{-1})x$. In particular, for all $x \in [0, 1]$, if $Y \sim \text{Pois}(x)$ or $Y = \sum_{i=1}^m \xi_i$, where ξ_1, \dots, ξ_m are independent Bernoulli r.v.'s with $\sum_{i=1}^m \mathbb{E}[\xi_i] = x$, then (by Lemma 3.8) $\mathbb{P}[Y \geq 1] \geq 1 - e^{-\mathbb{E}[Y]} \geq (1 - e^{-1})\mathbb{E}[Y]$.*

7 Tori - Proof of Theorem 4

We think of the vertices as being labeled by the set $[0, n-1]^d \cap \mathbb{Z}^d$. A **box** of side length r is a set of the form $\{(x_1, \dots, x_d) : \forall i, \exists j_i \in \{0, \dots, r-1\}, x_i \equiv v_i + j_i \pmod{n}\}$ for some $(v_1, \dots, v_d) \in [0, n-1]^d \cap \mathbb{Z}^d$. We define the ℓ_p distance ($p \geq 1$) between $x, y \in \mathbb{T}_d(n)$, $\|x - y\|_p$, as $\min \|x' - y'\|_p$, where the minimum is taken over all pairs x', y' in \mathbb{Z}^d such that $x' \equiv x$ and $y' \equiv y \pmod{n}$ (co-ordinate-wise) and $\|\cdot\|_p$ is the usual ℓ_p norm on \mathbb{R}^d .

Let $G = (V, E)$ be a graph and $Y = (Y_e)_{e \in E}$ be some indicator r.v.'s. We say that Y is **1-dependent** if $(Y_e)_{e \in E_1}$ and $(Y_e)_{e \in E_2}$ are independent, for all $E_1, E_2 \subset E$ such that $e_1 \cap e_2$ is empty (i.e. they do not share a common end-point), for all $(e_1, e_2) \in E_1 \times E_2$. We say that the random graph $(V, \{e \in E : Y_e = 1\})$ is 1-dependent if Y is 1-dependent.

7.1 An overview of the proof of Theorem 4.

For part (i) and $d = 2$ consider for simplicity the case in which there is one walker per site at time 0. The relative walk between two neighboring (at time 0) walkers is again a SRW and thus recurrent. Thus the 1-dependent random graph obtained by drawing an edge between two neighboring vertices iff the corresponding walks met by time t , has marginals arbitrarily close to 1 (if t is sufficiently large) and by a classical result [15] of Liggett et al. stochastically dominates super-critical independent percolation. This approach fails for $d > 2$. However, if one knew that the class of each walker grows locally sufficiently rapidly with a large probability, then the previous argument could be modified into a block argument. Namely, we consider a (1-dependent bond) percolation process on boxes of side length r (the boxes are thought of as vertices of a re-normalized torus of side length n/r), in which two

neighboring boxes are connected if the largest class of walkers at time r from each box has merged by time $t + r$, for some fixed $t > 0$. Fortunately, the desired estimate on the local growth of the model can be derived easily by the results of Kesten and Sidoravicius in [12].

As in § 6, we shall utilize part (i) in the proof of parts (ii)-(iii) by analyzing the time required for a “giant class” of walkers (as in (i)) to “capture” the rest of the walkers. As in § 6 we wish to obtain a quantitative estimate on the probability that a “giant class” captures a certain walker, conditioned on the walk performed by the walker, which is uniform over all possible walks. We shall derive such an estimate using the fact that (by comparison with independent percolation) the walkers of the “giant class” must have in some sense a positive spatial density (at least at sufficiently large scales).

Finally, the lower bound on SC is derived by establishing the existence of isolated walkers.

7.2 Proof of part (i) of Theorem 4

Fix some integer r to be determined later. Consider a partition of $\mathbb{T}_d(n)$ into $(\lfloor n/r \rfloor)^d$ boxes of side length r (apart from $O((n/r)^{d-1})$ boxes which may be of uneven side lengths between r and $2r$). The boxes inherit naturally the structure of $\mathbb{T}_d(\lfloor n/r \rfloor)$ (in fact, in what comes it suffices that they inherit the structure of a d -dimensional box of side length $\lfloor n/r \rfloor$). Namely, for every $v \in \mathbb{T}_d(\lfloor n/r \rfloor)$ we denote by B_v the unique box in the partition that contains rv . For every $v \in \mathbb{T}_d(\lfloor n/r \rfloor)$ let $\mathcal{W}(B_v)$ be the collection of walkers whose starting position is in B_v . For every v such that $|\mathcal{W}(B_v)| > 0$ we pick a single walker, w_v , from B_v . We pick w_v such that its starting position is the closest to the center of B_v in ℓ_1 distance (breaking ties arbitrarily).

We first treat the case $d = 2$. Consider the following auxiliary random subgraph of $\mathbb{T}_2(\lfloor n/r \rfloor)$ (a 1-dependent bond percolation) in which an edge is present between two neighboring vertices, v, u , if $|\mathcal{W}(B_v)| > 0$, $|\mathcal{W}(B_u)| > 0$ and w_v and w_u have met by time t (for some t to be determined shortly). Let $(\mathbf{w}_v(s))_{s \geq 0}$ (resp. $(\mathbf{w}_u(s))_{s \geq 0}$) be the walk performed by w_v (resp. w_u). Then $(\mathbf{w}_v(s) - \mathbf{w}_u(s))_{s \geq 0}$ is again a SRW (with a jump rate 2) and so by taking t and r to be sufficiently large we can make the probability of an edge $\{u, v\}$ being open become arbitrarily close to 1. Since the auxiliary percolation process we consider is 1-dependent by the results in [15] it follows that by taking r and t to be sufficiently large, it would stochastically dominate (super-critical) independent bond percolation on $\mathbb{T}_2(\lfloor n/r \rfloor)$ with parameter, say $3/4$. This concludes the proof for $d = 2$ using standard results in percolation theory (e.g. [16]).

We now turn to the case $d > 2$. Let $\mathcal{W}(v)$ be the walks in $\mathcal{W}(B_v)$ which are in the same class as w_v by time r . Let $B_v(r)$ be the collection of vertices in B_v which were visited by time r by some walker in $\mathcal{W}(v)$. By [12] we have that there exists some $c_1 = c_1(d)$ such that $|B_v(r)| \geq c_1 r^d$ w.p. tending to 1 as $r \rightarrow \infty$ (we formulated Theorem 4 in continuous-time to match the setup of [12]; We believe that both Theorem 4 and the results of [12] remain valid in the setup of discrete-time LSRWs). It is easy to deduce from this that there exists some $c_2 = c_2(d)$ such that $|\mathcal{W}(v)| \geq \ell := \lceil c_2 r^{d-1} / \log r \rceil$ w.p. tending to 1 as $r \rightarrow \infty$ (since the probability that some walker in $\mathcal{W}(B_v)$ has visited at least $8dr \log r$ distinct vertices by time r tends to 0 as $r \rightarrow \infty$). If (a) $|\mathcal{W}(v)| \geq \ell$ and in addition (b) the position of all of the walkers in $\mathcal{W}(v)$ at time r is at distance at most $2r$ from the center of B_v , we say that B_v is **good**. The probability that B_v is good tends to 1 as $r \rightarrow \infty$.

Consider the following auxiliary random subgraph of $\mathbb{T}_d(\lfloor n/r \rfloor)$ (a 1-dependent bond percolation) in which an edge is present between two neighboring vertices, v, u , if B_v and B_u are good and there is some pair of walkers $(w'_v, w'_u) \in \mathcal{W}(v) \times \mathcal{W}(u)$ that met by time t (for some t to be determined shortly). If B_v and B_u are both good, we can consider distinct $a_1, \dots, a_\ell \in \mathcal{W}(v)$, and $b_1, \dots, b_\ell \in \mathcal{W}(u)$. Let x_i and y_i be the positions of a_i and b_i , resp., at time r . As in the case $d = 2$, by considering the relative random walk between a_i and b_i , which is a SRW with jump rate 2, started at $x_i - y_i$ (we denote its law by $\mathbb{P}_{x_i - y_i}$) we get that the probability that they met between time r and $r + t$ is $\mathbb{P}_{x_i - y_i}[T_{\mathbf{0}} \leq t]$ (where $\mathbf{0} := (0, \dots, 0)$). Recall that for $d > 2$ the Green's function $G(x, y) := \sum_{i \geq 0} p^i(x, y)$ is proportional to $\|x - y\|_2^{2-d}$. For all $s \geq \|x - y\|_2^2$, $G(x, y)$ is proportional to both $G_s(x, y) := \sum_{i \leq s} p^i(x, y)$ (by the local CLT) and $\mathbb{P}_x[T_y < s]$ (by transience). Thus, by requirement (b) in the definition of a good box, the probability that a_i and b_i met between time r and $r + t$ for $t = C(dr)^2$, for some sufficiently large constant C , is at least $p_r := c_3(d)r^{2-d}$. Thus the probability that this fails for all $i \leq \ell$ is at most $(1 - p_r)^\ell \leq e^{-\ell p_r} \rightarrow 0$ as $r \rightarrow \infty$. As for $d = 2$, the proof is concluded using the main result in [15] to establish stochastic domination of super-critical bond percolation on $\mathbb{T}_d(\lfloor n/r \rfloor)$. \square

7.3 Proof of the upper bound on $\text{SC}(\mathbb{T}_d(n))$ from parts (ii)-(iii) of Theorem 4.

Lemma 7.1. *There exist some $C = C(d)$ and $c = c(d)$ such that w.h.p. in every box of side-length $L = L(d) := \lceil (C \log n)^{1/d} \rceil$ at time t_d (where t_d is as in part (i)) there are at least cL^d walkers belonging to the “giant class” from part (i).*

Proof. A corresponding statement is well-known in the setup of independent bond percolation (c.f. [16]). The assertion of the lemma follows using the stochastic domination utilized in the proof of part (i). More precisely, this implies a corresponding statement regarding the occupation measure of the good boxes. In practice, to translate this into a corresponding statement about the occupation measure of the walkers belonging to the “giant class”, one has to add the requirement that in order for an edge $\{u, v\}$ to be open in the auxiliary percolation process on the boxes, the positions of w_u and w_v at time t_d are at distance of at most, say $2t_d^2$, from their initial positions. \square

We denote the transition kernel for time t of continuous-time SRW on $\mathbb{T}_d(n)$ by $H_t(\cdot, \cdot)$.

Lemma 7.2. *Let $d \geq 2$. Fix some $C = C(d)$ and let $n \in \mathbb{N}$ be such that $L := \lceil (C \log n)^{1/d} \rceil \leq n^{1/d}$. Let $B \subset \mathbb{T}_d(n)$ be a box of side length L . Let $x \in B$. Then there exists some $c = c(d)$ such that for all $t \geq (9d \log d)L^2$ and all $y \in \mathbb{T}_d(n)$ such that $\|x - y\|_2 \leq 3\sqrt{t}$ we have that $H_t(x, y) \geq c \sum_{b \in B} \frac{1}{|B|} H_t(b, y)$.*

Proof. It suffices to show that $H_t(x, y) \geq cH_t(z, y)$ for all $z \in B$ and y such that $\|x - y\|_2 \leq 3\sqrt{t}$. This follows by the local CLT. We leave the details related to dealing with the finiteness of the graph to the reader (since $t \ll n^2$ we are justified in neglecting the effect of the finiteness of the graph). \square

Corollary 7.3. *There exist some $C = C(d)$ and $c_4 = c_4(d)$ such that w.h.p. (the probability is taken over the size of the giant class of time t_d , M_{t_d} , and over the positions at time t_d of the walkers belonging to the giant class) the following event, denoted by $G_{C,c}^{(n,d)}$, holds. Let*

$w_1, \dots, w_{M_{t_d}}$ be the walkers belonging to the “giant class” from part (i) (corresponding to time t_d). Denote the position of w_i at time t_d by v_i . Let $L := \lceil (C \log n)^{1/d} \rceil$. Then for every $t \geq (9d \log d)L^2$ we have that $\sum_{i=1}^{M_{t_d}} H_t(v_i, y) \geq c_4$ for all $y \in \mathbb{T}_d(n)$.

Proof. By combining the previous two lemmas, it is not hard to verify that it suffices to show that for all y we have that $\sum_{x: \|x-y\|_2 \leq 3\sqrt{t}} H_t(x, y) \geq c_5 > 0$. This follows by reversibility and the fact that for SRW on \mathbb{Z}^d we have that $H_t(y, \{x : \|x-y\|_2 \leq 3\sqrt{t}\}) \geq c_5$, for all t . \square

We are now in a position to conclude the proof of the upper bound on $\text{SC}(\mathbb{T}_d(n))$. We want to argue that the “giant class” captures each walker by time $t_d + s_d := t_d + C_d \log n$ for $d > 2$ and by time $t_2 + s_2 := t_2 + C_2 \log n \log \log n$ for $d = 2$ w.p. at least $1 - 1/n^{2d}$, regardless of the walk performed by that walker.

For all $d \geq 2$ and n , on the event $G_{C,c}^{(n,d)}$, which occurs w.h.p. by Corollary 7.3 (using its notation), there exists some $c_0 = c_0(d) > 0$ such that for every fixed walk $(\mathbf{w}(t))_{t \geq 0}$,

$$\sum_{i=1}^{M_{t_d}} \int_0^{s_d} H_t(v_i, \mathbf{w}(t_d + t)) dt \geq c_0 s_d \quad (7.1)$$

In order to conclude the proof, we need to transform this “expectation estimate” into a “probability estimate” indicating that on the event $G_{C,c}^{(n,d)}$, the probability that no walker from the giant class met a certain walker by time $t_d + s_d$ is exponentially small in s_d for $d > 2$ and in $s_2 / \log s_2$ for $d = 2$. Indeed, by the choice of s_d for $d \geq 2$ (and our freedom in the choice of C_d), such an estimate implies the aforementioned $1 - n^{-2d}$ estimate and thus concludes the proof of the upper bound on $\text{SC}(\mathbb{T}_d(n))$ from parts (ii)-(iii) of Theorem 4.

The following lemma allows us to do just that (think of the w_i ’s below as the walkers belonging to the giant class, of s as s_d as above, and so $\sum_i a_i$ below corresponds to the l.h.s. of (7.1) and hence is $\Omega(s_d)$). \square

Lemma 7.4. *Consider k independent walkers w_1, \dots, w_k whose initial positions are u_1, \dots, u_k . Consider an arbitrary RCLL (right continuous with left limits) path of vertices $\gamma := (v(t))_{t \in [0, \infty)}$. Let τ_i be the minimal t in which w_i is at $v(t)$ at time t . Let $2 \leq s \leq n$. Denote $a_i := \int_0^s H_t(u_i, v_i(t)) dt$ and $b_i := \mathbb{P}[\tau_i \leq s]$. Then there exists some c (independent of s , the u_i ’s and γ) such that*

(a) *If $d > 2$ for all i , $b_i \geq ca_i$. Hence, $\mathbb{P}[\min_i \tau_i > s] = \prod_i (1 - b_i) \leq \exp(-c \sum_i a_i)$.*

(b) *If $d = 2$, $b_i \geq \frac{ca_i}{\log s}$, for all i . Hence, $\mathbb{P}[\min_i \tau_i > s] = \prod_i (1 - b_i) \leq \exp(-\frac{c}{\log s} \sum_i a_i)$.*

Proof. It is standard that $\max_{x,y} H_t(x, y) = H_t(x, x) \leq C'(1+t)^{-d/2}$ for all t . In particular, for all $t \leq s$, $q_t := \int_t^s H_{t'-t}(v(t), v(t')) dt' \leq C \log s$ for $d = 2$ and $q_t \leq C$ for $d > 2$. So $a_i = \int_0^s f_i(t) q_t dt \leq C(1_{d>2} + 1_{d=2} \log s) \int_0^s f_i(t) dt = C(1_{d>2} + 1_{d=2} \log s) b_i$, where f_i is the density function of the law of τ_i . \square

7.4 Proof of the lower bound on $\text{SC}(\mathbb{T}_d(n))$ from parts (ii)-(iii) of Theorem 4.

Proof: The case $d > 2$ is covered by (1.2). Hence we only consider the case $d = 2$. Recall, we say that a walker remained **isolated** by time t if this walker did not meet any other walkers by time t (including). Our goal is to show that if $c > 0$ is sufficiently small, then w.h.p. there are walkers who remained isolated by time $r = r_n := \lfloor c \log n \log \log n \rfloor$. Fix some $v \in \mathbb{T}_2(n)$. As in the proof of (1.4), it suffices to show that (provided that c is sufficiently small) $\mathbb{P}[I_v] \geq c'n^{-\delta}$ for some $c' > 0$ and $\delta \in (0, 1)$, where I_v is the event that $Y_v(0) = 1$, i.e. at time 0 there is exactly one walker at vertex v , denoted by w_v , and that w_v remained isolated by time r . Throughout the proof below, we condition on the event that $Y_v(0) = 1$.

First consider the case that w_v did not leave v by time r . By Fact 2.1 the number of walkers that met w_v by time r would have a Poisson distribution with mean μ_r , where as in Remark 5.1, $\mu_r \leq 2r / \int_0^r H_t(v, v) dt \leq Cc \log n$, for some constant $C > 0$, independent of c . Hence if we pick $c < 1/(2C)$, the conditional probability that w_v remained isolated by time r (conditioned on staying put by time r) is indeed sufficiently large. We are not done yet, since the probability that w_v stayed put for r time units is not lower bounded by $\bar{c}n^{-\bar{\delta}}$ (for some $\bar{c}, \bar{\delta} > 0$). Our strategy is to show that, essentially, the same calculation is valid for a collection of walks, \mathcal{P} , such that the probability that w_v performs a walk from \mathcal{P} is at least $n^{-\alpha}$, for some $\alpha < 1$ such that $n^{-\alpha}e^{-\mu_r} \geq c'n^{-\beta}$ for some $\beta < 1$.

Let $\Gamma_t(v)$ be the collection of all RCLL walks $\gamma : [0, r] \rightarrow \mathbb{T}_2(n)$ such that $\gamma(0) = v$. Consider $\mathcal{P} := \{\gamma \in \Gamma_t(v) : \max_t \|\gamma(t) - v\|_\infty \leq (\log n)^{1/4}\}$. It is not hard to verify that the probability that the walker started from v performed by time r a walk in \mathcal{P} w.p. at least $\exp(-Cr/(\log^{1/4})^2) \geq \exp(-Cc\sqrt{\log n} \log \log n) = n^{-o(1)}$.

Finally, let $\gamma \in \mathcal{P}$. By Fact 2.1 the number of walkers that met w_v by time r , given that the walk that w_v performed by time r is $\gamma \in \mathcal{P}$, has a Poisson distribution with mean μ_γ . Extend γ to $[0, 2r]$ by setting $\gamma(t) = \gamma(r)$ for all $t \in (r, 2r]$. Let τ_u be the minimal t such that a (continuous-time, rate 1) SRW started from u is at $\gamma(t)$ at time t (if there is no such $t \leq 2r$ set $\tau_u = \infty$). Denote its density function by f_u . As in Remark 5.1,

$$2r \geq \sum_{u: u \neq v} \int_0^{2r} H_t(u, \gamma(t)) dt \geq \sum_{u: u \neq v} \int_0^r f_u(t) q_t dt,$$

where $q_t := \int_0^{2r-t} H_s(\gamma(t), \gamma(t+s)) ds$. As $\max_{t,t'} \|\gamma(t) - \gamma(t')\|_\infty \leq 2(\log n)^{1/4}$, it follows by the local CLT that for all sufficiently large n , for all $t \leq r$

$$q_t \geq c'_1 \int_{\sqrt{\log n}}^{2r-t} H_s(\gamma(t), \gamma(t)) ds \geq c'_2 \int_{\sqrt{\log n}}^r \frac{ds}{s} \geq c' \log \log n.$$

It follows that $\mu_\gamma = \sum_{u: u \neq v} \int_0^r f_u(t) dt \leq 2r/(c' \log \log n) \leq (c/c') \log n$. Thus if we set $c < c'/2$ we get that the conditional probability that the walker from v remained isolated by time r , given that she performed the walk γ is $e^{-\mu_\gamma} \geq n^{-1/2}$ as desired. \square

8 Concluding remarks, conjectures and open problems

The acquaintances graph $\text{AG}_t = (\mathcal{W}, E_t)$ at time t is the graph whose vertex set is the set of all walkers and whose edge set satisfies that $\{w, w'\} \in E_t$ iff w and w' have met by time

t . Note that $\text{SC} = \inf\{t : \text{AG}_t \text{ is connected}\}$. While for different underlying graphs AG_{SC} can have very different typical structures (e.g. when the underlying graph is an expander of size n we expect the order of the diameter of AG_{SC} to be between $\log n / \log \log n$ and $\log n$, while for the n -cycle it is between $n / \log n \log \log n$ and $n / \log n$) one feature of AG_{SC} shared by all bounded degree size n underlying graphs is that, w.h.p. the average and maximal degrees are polylogarithmic. In fact, in all of the (bounded degree) examples in this paper for which we determine the order of SC, the average (resp. maximal) degree in AG_{SC} is $\Theta(\log n)$ (resp. $O(\log^{3/2} n)$).

8.1 Some extensions of (1.1) beyond the LSRW setup

We now discuss some extensions (1.1). Our proof of (1.1), apart from the independence of the walks, used only the following properties of LSRW $(X_t)_{t \geq 0}$:

- (1) It is ergodic. Hence it has a stationary distribution, π . Consequently, when the density of walkers at each vertex is normalized so that it is proportional to π the expected number of walkers occupying a certain fixed vertex at time t does not depend on t .
- (2) The Markov property, along with the existence of a uniform decay (w.r.t. t) estimate for $\sup_{u,v \in V, s \geq 0} |\mathbb{P}[X_{t+s} = u \mid X_s = v] - \pi_u|$. This together with (1) is enough in order to establish that the configuration of walkers at each fixed time is δ -balanced t steps into the future w.p., say at least $1 - |V|^{-3}$, if $t = t_{|V|}$ is taken to be sufficiently large.
- (3) For all $x \in V$ and $s \geq 0$, for all sufficiently large t and neighboring $y, y' \in V$ both of distance at most $t/3$ from x

$$\mathbb{P}[X_{t+s} = y' \mid X_s = x] \geq c_d \mathbb{P}[X_{t+s} = y \mid X_s = x],$$

where d is the maximal degree. This is the only place in which we used the laziness of the walks.

Thus the proof could be extended to more general (but still independent) walks (they can even be time-inhomogeneous Markov chains, as long as we have (1)-(3)). It is easy to see that in order to derive a bound on SC for general walks (2) and (3) are both necessary. For (2) consider the case of continuous-time SRW with jump rate $1/t$. Clearly any bound on SC should depend on t . For (3), consider (non-lazy) weighted nearest neighbor random walk on some bipartite graph with one loop of weight δ added at a single vertex. Clearly, SC depends on δ .

Open Problem 8.1. *Assuming (1) and (3), find a more general condition than (2) which implies an upper bound on SC (without assuming the Markov property).*

It is not hard to see that the proof of (1.1) can also be extended to the following setup in which there are m “communities” of walkers. Let $m \leq n$. Let $k_1, \dots, k_m > 0$ such that $\sum_{i=1}^m k_i = n$. Let G_1, \dots, G_m be connected graphs on the same vertex set V of size n . Denote the stationary distribution of LSRW on G_i by π^i . At time 0 there are $\text{Pois}(k_i \pi_v^i)$ walkers at vertex i performing independent LSRW on G_i (for all $1 \leq i \leq m$ and $v \in V$, independently). As usual, when two walkers reach the same vertex at the same time they become acquainted (even if they walk on different graphs). The bound on SC depends in this case on $\max_{v \in V, i \leq m} d_v^i$, where d_v^i is the degree of v in G_i .

8.2 Dependence on the maximal degree and the vertex-transitive setup

Definition 8.2. We say that a bijection $\phi : V \rightarrow V$ is an automorphism of a graph $G = (V, E)$ if $\{u, v\} \in E$ iff $\{\phi(u), \phi(v)\} \in E$. A graph G is said to be vertex-transitive if the action of its automorphisms group, $\text{Aut}(G)$, on its vertices is transitive (i.e. $\{\phi(v) : \phi \in \text{Aut}(G)\} = V$ for all v).

Conjecture 8.3. There exists some $C > 0$ such that for every graph $G = (V, E)$ of maximal degree d , we have $\text{SC}(G) \leq C(d^2 1_{G \text{ is non-regular}} + d \cdot 1_{G \text{ is regular but not vertex-transitive}} + 1) \log^2 |V|$ w.p. at least $1 - C/|V|$.

Conjecture 8.4 (Monotonicity in the number of walkers). Denote by $\text{SC}(G, k)$ the social connectivity time when we have k walkers, each starting at a vertex chosen from the stationary distribution π independently of the rest of the walkers. Then for every vertex-transitive graph G , we have that $\mathbb{E}[\text{SC}(G, k)]$ is monotonically non-increasing in k for $k \geq 2$.

Remark 8.5. Consider the case that initially there are $\text{Poisson}(c|V| \log |V|)$ walkers. It is not hard to show that if $G = (V, E)$ has maximal degree d , then there exists some $M_{d,c}$ such that in the above setup $\text{SC}(G) \leq M_{d,c}$ w.h.p.. We believe $M_{d,c}$ can be taken to be independent of d .

Open Problem 8.6. Let $G = (V, E)$ be a finite connected vertex-transitive graph. Let $o \in V$. Consider the SN model on G . Consider

- $t(G)$, the minimal integer satisfying $t(G) \geq \log |V| \sum_{i=0}^{t(G)} p^i(o, o)$,
- τ_1 , the minimal time in which all classes are of size at least 2,
- and τ_2 , the minimal time by which every vertex has been visited by at least one walker.

Is there an absolute constant C such that

$$\max\{\mathbb{E}[\text{SC}(G)], \mathbb{E}[\tau_1], \mathbb{E}[\tau_2], t(G)\} \leq C \min\{\mathbb{E}[\text{SC}(G)], \mathbb{E}[\tau_1], \mathbb{E}[\tau_2], t(G)\} ?$$

Conjecture 8.7. Consider a sequence G_n of vertex-transitive graphs of increasing sizes. Then in the notation of Open problem 8.6,

$$\text{SC}(G_n)/\mathbb{E}[\text{SC}(G_n)] \rightarrow 1 \quad \text{and} \quad \tau_i(G_n)/\mathbb{E}[\tau_i(G_n)] \rightarrow 1 \quad (i = 1, 2) \quad \text{in probability, as } n \rightarrow \infty.$$

We believe that (in the setup of the previous conjecture) it is often the case that $[\text{SC}(G_n) - \tau_1(G_n)]/\mathbb{E}[\text{SC}(G_n)] \rightarrow 0$ in probability (e.g. we believe this is the case for transitive expanders). However, this might not be the case for the n -cycle. It is plausible that Conjecture 8.7 holds even without the transitivity assumption as long as the graphs G_n are of bounded degree. Conversely, if G_n is two disjoint n cliques connected by a single edge, then $\frac{\text{SC}(G_n)}{\mathbb{E}[\text{SC}(G_n)]}$ converges in distribution to the Exponential distribution with parameter 1.

In the last example we have that $\mathbb{E}[\text{SC}(G_n)] = \Theta(n)$. This shows $\mathbb{E}[\text{SC}(G_n)]$ may increase with the maximal degree. Below are two families of graphs, generalizing the previous example, which we believe to be extremal.

Remark 8.8. In general, $\mathbb{E}[\text{SC}(G)]$ may grow linearly in d , even when G is regular, as the following example demonstrates. Fix some $2 \leq d, n$ such that $2d \leq n$. Let J_k be a graph obtained from the complete graph on k vertices by deleting a single edge. Consider $\lceil n/d \rceil$ disjoint copies of J_d : $I_0, \dots, I_{\lceil n/d \rceil - 1}$, where for all $0 \leq j < \lceil n/d \rceil$, I_j is connected to I_{j+1} ($j+1$ is defined mod $\lceil n/d \rceil$) by a single edge that connects two degree $d-1$ vertices. This can be done so that the obtained graph is d -regular. It is easy to see that the obtained graph satisfies $\mathbb{E}[\text{SC}] \geq cd \log(n/d)$. We now consider a non-regular family of examples.

Take $\lceil n/\ell_{d,n} \rceil$ copies of J_{d-1} : $I_0, \dots, I_{\lceil n/\ell_{d,n} \rceil - 1}$ and for all $0 \leq k < \lceil n/\ell_{d,n} \rceil - 1$ connect I_k to I_{k+1} via a path of length $\ell_{d,n} := \lceil \frac{1}{4}d \log n \rceil$ (for some d such that $\lceil n/\ell_{d,n} \rceil \geq 2$) and attach to every vertex in $\cup_{k < \lceil n/\ell_{d,n} \rceil} I_k$ a path of length $\ell_{d,n}$. Then $\mathbb{E}[\text{SC}] \geq cd^2 \log^2 n = t_{d,n}$. To see this, observe that w.h.p. there will be many (length $\ell_{d,n}$) paths (ones have a degree 1 end-point) which will be occupied initially by a single walker that up to time $t_{d,n}$ will remain within distance $\ell_{d,n}/2$ from the closest copy of J_d , with no other walker crossing the middle point of that vertex by that time. We leave the details to the reader (hint: use the idea from Remark 5.1).

8.3 Additional Open Problems

The following conjecture concerns the existence of infinite classes of walkers after a constant number of steps in the setup of infinite graphs.

Conjecture 8.9. For every infinite bounded degree graph G with $p_c(G) < 1$ (i.e. the critical density for independent percolation on G is less than 1), if the number of walkers at time 0 at each $v \sim \text{Pois}(\lambda d_v)$, independently for different v 's (let \mathbb{P}_λ be the corresponding probability measure), then for every $\lambda > 0$ there exists $t_c(\lambda, G) > 0$ such that for all $t > t_c(\lambda, G)$,

$$\mathbb{P}_\lambda[\text{there exists an infinite class of walkers at time } t] = 1.$$

Remark 8.10. It is not hard to extend the analysis from the proof of Theorem 4 to show that the conjecture holds when $G = \mathbb{Z}^d$ (at least in the continuous-time setup). The case that G is non-amenable is treated in [9, Theorem 3].

Let K_n be the complete graph on n vertices. When the holding probabilities of the walks are taken to be either $q_n = 0$ or $q_n = 1/n$, one can show that after a single step the SN model (with $\text{Pois}(1)$ walkers per site) behaves in some sense like a critical Erdős Rényi random graph, $G(n, 1/n)$, while after two steps it behaves like the super-critical random graph $G(n, 2/n)$ (hence w.h.p. a giant class of walkers emerges after precisely 2 steps). We believe that also in the continuous-time setup there is a phase transition.

Open Problem 8.11. Consider the SN model on K_n in continuous-time (each walker has jump rate 1). What is the critical time, t_c , for the emergence of a giant class of walkers?

Acknowledgements

We thank Gady Kozma for his great contribution for some of the results presented in this paper. We like to thank also Riddhipratim Basu, Hilary Finucane, Eviatar Procaccia, Allan Sly and especially Ofer Zeitouni for many useful discussions and for reading some version of this work carefully and providing helpful comments.

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